XIV

ON THE FUNDAMENTAL EQUATIONS OF ELECTROMAGNETICS FOR BODIES IN MOTION

(*Wiedemann's Ann.* **41**, p. 369, 1890)

AN account which I recently published[1] of electromagnetic processes in bodies at rest agreed, as far as the matter was concerned, with Maxwell's theory, but as far as the manner was concerned it aimed at a more systematic arrangement. From the outset the conception was insisted upon, that the electric and magnetic forces at any point owe their action to the particular condition of the medium which fills the space at that point; and that the causes which determine the existence and variations of these conditions are to be wholly sought in the conditions of the immediate neighbourhood, excluding all actions-at-a-distance. It was further assumed that the electric and magnetic state of the medium which fills space could be completely determined for every point by a single directed magnitude for each state; and it was shown that the restriction which lies in this assumption only excluded from consideration comparatively unimportant phenomena. The introduction of potentials into the fundamental equations was avoided.

The question now arises whether, while adhering strictly to the same views and the same limitations, the theory can be extended so as to embrace the course of electromagnetic phenomena in bodies which are in motion. We remark, in the first place, that whenever in ordinary speech we speak of bodies in motion, we have in mind the motion of ponderable matter alone. According to our view, however, the disturbances of the ether, which simultaneously arise, cannot be without effect;

[1] See XIII. p. 195.

and of these we have no knowledge. This is equivalent to saying that the question here raised cannot at present be treated at all without introducing arbitrary assumptions as to the motion of the ether. Furthermore, the few existing indications as to the nature of the motion of the ether lead us to suppose that the question above raised is strictly to be answered in the negative, for it appears to follow from such indications as we have, that even in the interior of tangible matter the ether moves independently of it; indeed, this view can scarcely be avoided in view of the fact that we cannot remove the ether from any closed space. If now we wish to adapt our theory to this view, we have to regard the electromagnetic conditions of the ether and of the tangible matter at every point in space as being in a certain sense independent of each other. Electromagnetic phenomena in bodies in motion would then belong to that class of phenomena which cannot be satisfactorily treated without the introduction of at least two directed magnitudes for the electric and two for the magnetic state.

But the state of the case is different if we explicitly content ourselves with representing electromagnetic phenomena in a narrower sense—up to the extent to which they have hitherto been satisfactorily investigated. We may assert that among the phenomena so embraced there is not one which requires the admission of a motion of the ether independently of ponderable matter within this latter; this follows at once from the fact that from this class of phenomena no hint has been obtained as to the magnitude of the relative displacement. At least this class of electric and magnetic phenomena must be compatible with the view that no such displacement occurs, but that the ether which is hypothetically assumed to exist in the interior of ponderable matter only moves with it. This view includes the possibility of taking into consideration at every point in space the condition of only one medium filling the space; and it thus admits of the question being answered in the affirmative. For the purpose of the present paper we adopt this view. It is true that a theory built on such a foundation will not possess the advantage of giving to every question that may be raised the correct answer, or even of giving only one definite answer; but it at least gives possible answers to every question that may be propounded, *i.e.* answers which are not inconsistent with the observed phenomena nor yet with the views which we have obtained as to bodies at rest.

We therefore assume that at every point a single definite velocity can be assigned to the medium which fills space; and we denote the components of this in the directions of x, y, z by α, β, γ. We regard these magnitudes as being everywhere finite, and treat them as varying continuously from point to point. Of course we also admit discontinuous variations, but we regard them as being only the limiting cases of very rapid continuous variations. We further limit each permissible discontinuity by the restriction that it shall in no case lead to the formation of empty spaces. The necessary and sufficient condition for this is that the three differential coefficients $d\alpha/dx$, $d\beta/dy$, $d\gamma/dz$ should everywhere be finite. Wherever we find tangible matter in space we definitely deduce the values of α, β, γ from the motion of this. Wherever we do not find in the space any tangible matter, we may assign to α, β, γ any arbitrary value which is consistent with the given motions at the boundary of the empty space, and is of the same order of magnitude. We might, for example, give α, β, γ those values which would exist in the ether if it moved like any gas. We further use all the symbols which occur in the preceding paper in the same sense here. We here regard electric and magnetic force as signs of the condition of the moving matter in the same sense in which we have hitherto regarded them as signs of the conditions of matter at rest. Electric and magnetic polarisation we simply regard as a second and equivalent means of indicating the same conditions. We also assign to the lines of force, by which we represent these polarisations, precisely the same meaning.

1. *Statement of the Fundamental Equations for Bodies in Motion*

At any point of a body at rest the time-variation of the magnetic state is determined simply by the distribution of the electric force in the neighbourhood of the point. In the case of a body in motion there is, in addition to this, a second variation which at every instant is superposed upon the first, and which arises from the distortion which the neighbourhood of the point under consideration experiences through the motion. We now assert that the influence of the motion is of such a kind that, if it alone were at work, it would carry the magnetic lines of force with the matter. Or more precisely:—Supposing that at any given instant the magnetic state of the substance was represented in magnitude and direction by a system of lines of force; then a system of lines of force passing through the same material points would also represent in magnitude and direction the magnetic state at any other time, if the effect of the motion alone had to be considered. The corresponding statement holds good for the variation which the electric polarisation experiences through the motion. These statements suffice for extending to moving bodies the theory already developed for bodies at rest; they clearly satisfy the conditions which our system of itself requires, and it will be shown that they embrace all the observed facts.[1]

In order to represent our ideas symbolically, let us first, during the time-element dt, fix our attention upon a small surface-element in the interior of the moving matter, which at the beginning of this time-element lies parallel to the yz-plane, and during the motion is displaced and distorted with the matter. We distribute and draw the magnetic lines of force so that the number of them which penetrates the surface-element at the beginning of the time dt is $\mathfrak{L}$. Everywhere and always $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ will then denote the number of lines of force which traverse a surface-element of equal area parallel to the yz, xz, xy-planes respectively. The number of lines of force which traverses our particular surface-element now varies owing to several causes; we shall consider separately the amount which each separate cause contributes. In the first place, the number would vary even if the surface-element remained in its original position; this variation amounts to $(d\mathfrak{L}/dt)dt$, if $d\mathfrak{L}/dt$ denotes the rate of variation of $\mathfrak{L}$ at a point which, with reference to our system of co-ordinates, is at rest. In the second place, since the surface-element is displaced with the velocity α, β, γ to places where other values

[1] [See Note 33 at end of book.]

of $\mathfrak{L}$ obtain, the rate of variation due to this cause amounts to $(\alpha d\mathfrak{L}/dx + \beta d\mathfrak{L}/dy + \gamma d\mathfrak{L}/dz)dt$. In the third place, the plane of the element rotates with velocity $d\alpha/dy$ about the z-axis, and with velocity $d\alpha/dz$ about the y-axis, and lines of force will be embraced by the element which originally were parallel to it; the amount due to this cause is—$(\mathfrak{M}d\alpha/dy + \mathfrak{N}d\alpha/dz)dt$. Finally the surface of the element increases with velocity $d\beta/dy + d\gamma/dz$; and for this cause the number increases by an amount $\mathfrak{L}(d\beta/dy + d\gamma/dz)dt$. If the sum of these quantities is equal to zero, there can be no change in the number; we have therefore reckoned up completely all causes of variation, and since all the amounts are very small, their sum represents the total variation. We may also analyse the total variation in another manner which has a more distinct physical significance, viz. into the amount which the presence of the electric forces in the neighbourhood, and the amount which the motion would contribute, each by itself and in the supposed absence of the other cause. According to the laws which hold good for conductors at rest, the first amounts to $(dZ/dy - dY/dz)dt \,.\, 1/A$; according to the statement which we have just made, the latter is zero; the first of itself represents the total variation. We equate the two expressions found for the total variation, divide by dt, multiply by A, add and subtract the terms $\alpha d\mathfrak{M}/dy + \alpha d\mathfrak{N}/dz$, rearrange the terms and thus obtain, after treating similarly the other components of the magnetic force and the components of the electric force, the following system of fundamental equations for bodies in motion:—

$$(1_a)\quad \left\{\begin{aligned}
&A\left\{\frac{d\mathfrak{L}}{dt} + \frac{d}{dy}(\beta\mathfrak{L} - \alpha\mathfrak{M}) - \frac{d}{dz}(\alpha\mathfrak{N} - \gamma\mathfrak{L}) + \alpha\left(\frac{d\mathfrak{L}}{dx} + \frac{d\mathfrak{M}}{dy} + \frac{d\mathfrak{N}}{dz}\right)\right\} \\
&\qquad\qquad = \frac{dZ}{dy} - \frac{dY}{dz}, \\
&A\left\{\frac{d\mathfrak{M}}{dt} + \frac{d}{dz}(\gamma\mathfrak{M} - \beta\mathfrak{N}) - \frac{d}{dx}(\beta\mathfrak{L} - \alpha\mathfrak{M}) + \beta\left(\frac{d\mathfrak{L}}{dx} + \frac{d\mathfrak{M}}{dy} + \frac{d\mathfrak{N}}{dz}\right)\right\} \\
&\qquad\qquad = \frac{dX}{dz} - \frac{dZ}{dx}, \\
&A\left\{\frac{d\mathfrak{N}}{dt} + \frac{d}{dx}(\alpha\mathfrak{N} - \gamma\mathfrak{L}) - \frac{d}{dy}(\gamma\mathfrak{M} - \beta\mathfrak{N}) + \gamma\left(\frac{d\mathfrak{L}}{dx} + \frac{d\mathfrak{M}}{dy} + \frac{d\mathfrak{N}}{dz}\right)\right\} \\
&\qquad\qquad = \frac{dY}{dx} - \frac{dX}{dy},
\end{aligned}\right.$$

$$
(1_b)\left\{
\begin{aligned}
&A\left\{\frac{d\mathfrak{X}}{dt}+\frac{d}{dy}(\beta\mathfrak{X}-a\mathfrak{Y})-\frac{d}{dz}(a\mathfrak{Z}-\gamma\mathfrak{X})+a\left(\frac{d\mathfrak{X}}{dx}+\frac{d\mathfrak{Y}}{dy}+\frac{d\mathfrak{Z}}{dz}\right)\right\}\\
&\qquad=\frac{dM}{dz}-\frac{dN}{dy}-4\pi Au,\\
&A\left\{\frac{d\mathfrak{Y}}{dt}+\frac{d}{dz}(\gamma\mathfrak{Y}-\beta\mathfrak{Z})-\frac{d}{dx}(\beta\mathfrak{X}-a\mathfrak{Y})+\beta\left(\frac{d\mathfrak{X}}{dx}+\frac{d\mathfrak{Y}}{dy}+\frac{d\mathfrak{Z}}{dz}\right)\right\}\\
&\qquad=\frac{dN}{dx}-\frac{dL}{dz}-4\pi Av,\\
&A\left\{\frac{d\mathfrak{Z}}{dt}+\frac{d}{dx}(a\mathfrak{Z}-\gamma\mathfrak{X})-\frac{d}{dy}(\gamma\mathfrak{Y}-\beta\mathfrak{Z})+\gamma\left(\frac{d\mathfrak{X}}{dx}+\frac{d\mathfrak{Y}}{dy}+\frac{d\mathfrak{Z}}{dz}\right)\right\}\\
&\qquad=\frac{dL}{dy}-\frac{dM}{dx}4\pi Aw.
\end{aligned}\right.
$$

which are completed by the linear relations which connect the polarisations and the current-components with the forces. The constants of these relations are to be regarded as functions of the varying conditions of the moving matter, and to this extent as functions of the time as well.[1]

Our method of deducing the equations (1_a) and (1_b) does not require that the system of co-ordinates used should remain absolutely fixed in space. We can, therefore, without change of form, transform our equations from the system of co-ordinates first chosen to a system of co-ordinates moving in any manner through space, by taking a, β, γ to represent the velocity-components with reference to the new system of co-ordinates, and referring the constants ϵ, μ, λ, X', Y', Z', which depend upon direction, at every instant to these. From this it follows that the absolute motion of a rigid system of bodies has no effect upon any internal electromagnetic processes whatever in it, provided that *all* the bodies under consideration, including the ether as well, actually share the motion. It further follows from this consideration that even if only a single part of a moving system moves as a rigid body, the processes which occur in this part follow exactly the same course as in bodies at rest. If, nevertheless, the existing motion does exert any influence upon this part, this influence can only arise in those portions of the system in which distortion of the elements occurs, and must be propagated thence into those portions which move

[1] [See Note 34 at end of book.]

after the manner of rigid bodies. If, for example, a solid mass of metal is suddenly displaced in the magnetic field, then, according to our equations, the only direct or simultaneous effect of this disturbance is upon the surface and the neighbourhood of the metallic mass; it here gives rise to electric forces which afterwards produce secondary effects—penetrating into the interior of the mass and giving rise to currents in it.

The equations here stated are in form and intention closely related to those by which von Helmholtz in vol. lxxviii. of Borchardt's *Journal* represented the behaviour of the electric and magnetic forces in moving bodies.[1] From this source the notation is partly borrowed. And yet our equations differ from those given by v. Helmholtz not only in form, but also in meaning, at least with regard to such members as have not hitherto been tested by experiment. Maxwell himself does not seem to me to have aimed in his treatise at any systematic treatment of the phenomena in moving bodies.[2] The numerous references which he makes to such phenomena are either confined to approximations, or relate only to cases which do not involve any necessary distinction between the theories of direct and of indirect action.

2. *The Physical Meaning of the Separate Terms*

Equations (1_a) and (1_b) tell us the future value of the polarisations at every fixed point in space or, if we prefer it, in each element of the moving matter, as a definite and determinate consequence of the present electromagnetic state and the present motion in the neighbourhood of the point under consideration. This is the physical meaning of them in accordance with the conception which our system represents. The common conception of the relations expressed by these equations is quite different. It regards the rates of variation of the polarisations on the left-hand side as the cause, and the induced forces on the right-hand side as the

[1] v. Helmholtz, *Ges. Abhandl.* **1**, p. 745; Borchardt's *Journ. f. Mathem.* **78**, p. 273, 1874.

[2] [This statement is not quite correct. It does indeed hold good for Maxwell's treatise, to which it refers; but in his paper "On Physical Lines of Force" (*Phil. Mag.*, April 1861) Maxwell has himself given a complete and systematic treatment of the phenomena in moving bodies. Unfortunately I had not noticed this when writing my paper.]

consequence thereof. This conception has arisen through the fact that the polarisations and their variations are usually sooner and more clearly known to us than the forces which simultaneously arise; so that, as far as our knowledge goes, the left-hand sides of the equations are prior to the right-hand sides. In the cases which chiefly interest us this conception has indeed very great advantages; but from the general standpoint it has the disadvantage that the forces are not definitely determined by the rates of variation of the polarisations of the opposite kind, but contain terms which are independent of these variations. The common theory gets out of this difficulty by setting these terms as electrostatic or magnetic forces in opposition to the electromagnetic forces which are alone, according to that theory determined by our equations. Although we do not approve of such a separation, and hence do not accept the common conception as to the causal relationship, it is still interesting to show how the partial forces which are introduced in the usual theory are contained in the separate terms of our equations. For this purpose we split up the forces in the form $X = X_1 + X_2$, etc., $L = L_1 + L_2$, etc., and put—

$$(2)\begin{cases} X_1 = A(\gamma\mathfrak{M} - \beta\mathfrak{N}), & L_1 = A(\beta\mathfrak{Z} - \gamma\mathfrak{Y}), \\ Y_1 = A(\alpha\mathfrak{N} - \gamma\mathfrak{L}), & M_1 = A(\gamma\mathfrak{X} - \alpha\mathfrak{Z}), \\ Z_1 = A(\beta\mathfrak{L} - \alpha\mathfrak{M}), & N_1 = A(\alpha\mathfrak{Y} - \beta\mathfrak{X}), \end{cases}$$

We thus obtain for X_2, Y_2, Z_2, L_2, M_2, N_2 equations which result from the equations (1_a) and (1_b) for X, Y, Z, L, M, N by omitting the second and third terms on the left-hand side. Now the resultant of X_1 Y_1 Z_1 is an electric force which arises as soon as a body moves in the magnetic field. It is perpendicular to the direction of the motion and to the direction of the magnetic lines of force; it is that force which in a narrower sense we are accustomed to denote as the electromotive force induced through motion. But it should be observed that, according to our views, the separation of this from the total force can have no physical meaning; for it would be in opposition to our conception to suppose that the magnetic field within a body could have a motion relative to it. The counterpart to the force X_1 Y_1 Z_1 is the force L_1 M_1 N_1, which must make itself felt in a non-conductor when the latter is displaced through the lines of force of an electric field; but this is not yet confirmed by experience and is absent from the older electromagnetics.

Let us now turn our attention to the resultant of $L_2\ M_2\ N_2$, and suppose the general solutions of the equations containing these quantities to be represented as functions of the quantities

$$u,\ d\mathfrak{X}/dt,\ a(d\mathfrak{X}/dx + d\mathfrak{Y}/dy + d\mathfrak{Z}/dz),\ \text{etc.}$$

Let us put these latter quantities in the functions all equal to zero; there still remains a first part of the force which does not owe its origin to electromagnetic causes. Its components necessarily possess a potential; it represents that distance-force which, according to the older view, proceeds from magnetic masses. A second part of the force is given by that part of the functions which vanishes when, and only when, u, v, w vanish. It contains the magnetic distance-force which appears to proceed from the actual electric currents. We obtain the whole of the electromagnetic part of the force $L_2\ M_2\ N_2$ by replacing in the expression of the second part the quantity $4\pi Au$ by the quantity

$$4\pi Au + A\frac{d\mathfrak{X}}{dt} + Aa\left(\frac{d\mathfrak{X}}{dx} + \frac{d\mathfrak{Y}}{dy} + \frac{d\mathfrak{Z}}{dz}\right)$$

and treating v and w similarly. This corresponds to the statement that as far as the production of a magnetic distance-force is concerned, an actual current is to be regarded as equivalent in the first place to the variation of an electric polarisation, and in the second place to the convective motion of true electricity. The latter part of this statement finds its requisite confirmation in Rowland's experiment.

Finally, let us consider the force $X_2\ Y_2\ Z_2$. We can separate from this force as well a part which is independent of time-variations of the system, which possesses a potential, and which is treated as an electrostatic distance-force. From the residue of the electromagnetic force which remains we can detach a second part, which vanishes when, and only when, the quantities $d\mathfrak{L}/dt$, $d\mathfrak{M}/dt$, $d\mathfrak{N}/dt$ vanish. It clearly represents the force of induction which arises from varying magnetic moments, but it also contains in a hidden form that electric force which owes its origin to varying currents.

Finally, there remains a third and last part which can be interpreted as an electric force produced by a convective motion of magnetism, and in which must be found the explanation of certain known phenomena of unipolar induction.

These considerations show that we might also have arrived at the system of equations (1_a) and (1_b) by summing up the effects of the separate forces required by the older theories, and adding a series of hypothetical terms which can neither be confirmed nor disproved by existing experience. The way which we have followed requires a smaller number of independent hypotheses. We now proceed to deduce from our equations the most important general results.

3. *Motion of Magnets and of Electrostatically Charged Bodies*

As independent causes of variation of the electric or magnetic polarisation there appear in our scheme first the magnetic or electric forces respectively, and secondly the motion of material bodies. According to our conclusions in the case of bodies at rest, the first cause produces no displacement of true electricity in non-conductors and no displacement of true magnetism at all. The latter cause of itself produces a displacement of electricity and of magnetism towards the space at rest, but it can cause no displacement towards the matter in motion; for by its motion this matter carries with it the lines of force, and electricity and magnetism may be regarded as the free ends of these lines. Hence when both causes act together there can be no relative motion of true magnetism with reference to the surrounding matter; nor can there be any such relative motion of true electricity, at any rate in non-conductors. Under these circumstances electricity and magnetism move with the matter in which they are present, as if they were indestructible and adhered firmly to the parts thereof. In order to represent this same idea symbolically, let us differentiate first the equations (1_a) and then the equations (1_b) with respect to $x\,y\,z$, multiply by the volume-element $d\tau$ which we suppose to remain at rest, and to which the quantities $\mathfrak{L}$, $\mathfrak{M}$, etc., refer. Let $d\tau'$ denote a volume-element which at every instant encloses the matter contained at the present instant in $d\tau$; let de' and dm' denote the amounts of true electricity and true magnetism respectively contained in $d\tau'$, and $\mathfrak{L}'$, $\mathfrak{M}'$, etc., the values of $\mathfrak{L}$, $\mathfrak{M}$, etc., with reference to $d\tau'$. We thus obtain—

$$(3_a)\quad \begin{cases} \Big\{\dfrac{d}{dt}\Big(\dfrac{d\mathfrak{L}}{dx}+\dfrac{d\mathfrak{M}}{dy}+\dfrac{d\mathfrak{N}}{dz}\Big) \\ \quad +\alpha\dfrac{d}{dx}\Big(\dfrac{d\mathfrak{L}}{dx}+\dfrac{d\mathfrak{M}}{dy}+\dfrac{d\mathfrak{N}}{dz}\Big)+\beta\dfrac{d}{dy}\Big(\dfrac{d\mathfrak{L}}{dx}+\dfrac{d\mathfrak{M}}{dy}+\dfrac{d\mathfrak{N}}{dz}\Big) \\ \qquad +\gamma\dfrac{d}{dz}\Big(\dfrac{d\mathfrak{L}}{dx}+\dfrac{d\mathfrak{M}}{dy}+\dfrac{d\mathfrak{N}}{dz}\Big) \\ \quad +\Big(\dfrac{d\alpha}{dx}+\dfrac{d\beta}{dy}+\dfrac{d\gamma}{dz}\Big)\Big(\dfrac{d\mathfrak{L}}{dx}+\dfrac{d\mathfrak{M}}{dy}+\dfrac{d\mathfrak{N}}{dz}\Big)\Big\}d\tau \\ \quad =\dfrac{d}{dt}\Big\{\Big(\dfrac{d\mathfrak{L}'}{dx}+\dfrac{d\mathfrak{M}'}{dy}+\dfrac{d\mathfrak{N}'}{dz}\Big)d\tau'\Big\}=4\pi\dfrac{dm'}{dt}=0, \end{cases}$$

$$(3_b)\quad \begin{cases} \Big\{\dfrac{d}{dt}\Big(\dfrac{d\mathfrak{X}}{dx}+\dfrac{d\mathfrak{Y}}{dy}+\dfrac{d\mathfrak{Z}}{dz}\Big) \\ \quad +\alpha\dfrac{d}{dx}\Big(\dfrac{d\mathfrak{X}}{dx}+\dfrac{d\mathfrak{Y}}{dy}+\dfrac{d\mathfrak{Z}}{dz}\Big)+\beta\dfrac{d}{dy}\Big(\dfrac{d\mathfrak{X}}{dx}+\dfrac{d\mathfrak{Y}}{dy}+\dfrac{d\mathfrak{Z}}{dz}\Big) \\ \qquad +\gamma\dfrac{d}{dz}\Big(\dfrac{d\mathfrak{X}}{dx}+\dfrac{d\mathfrak{Y}}{dy}+\dfrac{d\mathfrak{Z}}{dz}\Big) \\ \quad +\Big(\dfrac{d\alpha}{dx}+\dfrac{d\beta}{dy}+\dfrac{d\gamma}{dz}\Big)\Big(\dfrac{d\mathfrak{X}}{dx}+\dfrac{d\mathfrak{Y}}{dy}+\dfrac{d\mathfrak{Z}}{dz}\Big)\Big\}d\tau \\ \quad =\dfrac{d}{dt}\Big\{\Big(\dfrac{d\mathfrak{X}'}{dx}+\dfrac{d\mathfrak{Y}'}{dy}+\dfrac{d\mathfrak{Z}'}{dz}\Big)d\tau'\Big\}=4\pi\dfrac{de}{dt} \\ \quad =4\pi\Big(\dfrac{du}{dx}+\dfrac{dv}{dy}+\dfrac{dw}{dz}\Big). \end{cases}$$

These equations embrace the statements already made, and complete them as far as conductors are concerned. If the velocities $\alpha\ \beta\ \gamma$ are so small that the electric and magnetic conditions may at each instant remain infinitely near to the stationary state, and if we restrict ourselves to the consideration of such quasi-stationary states, then the results which we have obtained are sufficient and necessary to determine the interdependence of the various states which may arise from each other. The introduction of these results into such problems enables us to replace the complete, but very complicated, equations (1_a) and (1_b) by the equivalent and very simple equations which hold good for statical problems in bodies at rest, and which can be deduced from equations (1_a) and (1_b) by equating to zero the velocities and the time-variations at all points of the space. Such a simplification of the statements is not possible without introducing the idea of electricity and of magnetism; and it seems to me that this is the principal reason why these ideas are indispensable in the study of electrostatics and in the representation of magnetic phenomena.

4. *Induction in Closed Circuits*

The greatest velocities which we can assign to the surrounding bodies are so small compared with the velocity of light—the reciprocal of which appears as the multiplier of α, β, γ in equations (1_a) and (1_b)—that electromagnetic effects due purely to motion can only be investigated with precision in the particular case in which these effects consist in the induction of an electric current in a closed metallic conductor. In order to determine the magnitude of such effects in closed conductors, let us consider any unclosed portion ω of a surface in the interior of the matter under consideration, and which is displaced with the material particles during the motion. Let s represent the instantaneous limiting curve of this surface-element. Let ζ' denote the number of magnetic lines of force which at any time traverse the surface ω. We shall again consider the causes which produce (independently of each other) a variation in ζ' as being two—in the first place, the electric forces; and in the second place, the motion of matter. If the first cause alone were at work, the system would be at rest, and so the rate of variation of ζ' multiplied by A would be equal to the integral of the electric force taken around the whole extent of s; the integral, viewed from the side of the positive normal, being taken clock-wise. If the motion alone were at work, it would not produce any variation of ζ', for it would carry forward the lines of force traversing the surface ω together with this surface itself. Hence in the actual case in which the two causes act together, the integral of the electric force taken in the given sense around any closed curve s is equal to A multiplied by the rate of variation of the number of magnetic lines of force which traverse any surface which was originally bounded by the curve s, but which follows the motion. This law also holds good for the special case—the only one which is important from an experimental point of view—in which the curve s follows the path of a linear conductor; nor does it become invalid when the motion is sufficiently slow to allow all the states which arise to appear as being steady, and the current as uniform in all parts of the conductor.

To represent this symbolically, let n',x, n',y, n',z denote the angle which the normal to the element $d\acute{\omega}$ of the moving surface $\acute{\omega}$ makes at any instant with the axes. Let $\mathfrak{L}'\ \mathfrak{M}'\ \mathfrak{N}'$ be the values of $\mathfrak{L}\ \mathfrak{M}\ \mathfrak{N}$ in this element. Further, let $d\omega$, n,x, n,y, n,z denote the values of $d\acute{\omega}$, n',x, n',y, n',z in the original position. We observe that, from purely geometrical considerations, we have

$$\frac{d}{dt}(d\acute{\omega}\cos n',x) = d\omega\left\{\left(\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right)\cos n,x - \frac{d\beta}{dx}\cos n,y - \frac{d\gamma}{dx}\cos n,z\right\},$$

$$\frac{d}{dt}(d\acute{\omega}\cos n',y) = d\omega\left\{-\frac{d\alpha}{dy}\cos n,x + \left(\frac{d\alpha}{dx}+\frac{d\gamma}{dz}\right)\cos n,y - \frac{d\gamma}{dy}\cos n,z\right\},$$

$$\frac{d}{dt}(d\acute{\omega}\cos n',z) = d\omega\left\{-\frac{d\alpha}{dz}\cos n,x - \frac{d\beta}{dz}\cos n,y + \left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}\right)\cos n,z\right\},$$

and we thus obtain

$$\frac{d\zeta'}{dt} = \frac{d}{dt}\int(\mathfrak{L}'\cos n',x + \mathfrak{M}'\cos n',y + \mathfrak{N}'\cos n',z)d\acute{\omega}$$

$$= \int\left(\frac{d\mathfrak{L}}{dt}+\alpha\frac{d\mathfrak{L}}{dx}+\beta\frac{d\mathfrak{L}}{dy}+\gamma\frac{d\mathfrak{L}}{dz}\right)\cos n,x\,d\omega$$

$$+\int\left(\frac{d\mathfrak{M}}{dt}+\alpha\frac{d\mathfrak{M}}{dx}+\beta\frac{d\mathfrak{M}}{dy}+\gamma\frac{d\mathfrak{M}}{dz}\right)\cos n,y\,d\omega$$

$$+\int\left(\frac{d\mathfrak{N}}{dt}+\alpha\frac{d\mathfrak{N}}{dx}+\beta\frac{d\mathfrak{N}}{dy}+\gamma\frac{d\mathfrak{N}}{dz}\right)\cos n,z\,d\omega$$

$$+\int\mathfrak{L}\left(\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right)\cos n,x\,d\omega - \int\mathfrak{L}\frac{d\beta}{dx}\cos n,y\,d\omega - \int\mathfrak{L}\frac{d\gamma}{dx}\cos n,z\,d\omega$$

$$-\int\mathfrak{M}\frac{d\alpha}{dy}\cos n,x\,d\omega - \int\mathfrak{M}\left(\frac{d\alpha}{dx}+\frac{d\gamma}{dz}\right)\cos n,y\,d\omega - \int\mathfrak{M}\frac{d\gamma}{dy}\cos n,z\,d\omega$$

$$-\int\mathfrak{N}\frac{d\alpha}{dz}\cos n,x\,d\omega - \int\mathfrak{N}\frac{d\beta}{dz}\cos n,y\,d\omega + \int\mathfrak{N}\left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}\right)\cos n,z\,d\omega\,;$$

from which, with the aid of equations (1_a) and (1_b) we get

$$A\frac{d\zeta'}{dt}=\int\left\{\left(\frac{dZ}{dy}-\frac{dY}{dz}\right)\cos n,x+\left(\frac{dX}{dz}-\frac{dZ}{dx}\right)\cos n,y+\left(\frac{dY}{dx}-\frac{dX}{dy}\right)\cos n,z\right\}d\omega$$
$$=\int(Xdx+Ydy+Zdz),$$

the last integral being taken around the whole extent s of the surface $d\omega$.

In special cases these results admit of simplification. If it is possible to shut off a singly connected space which entirely contains the moving curve s, and in which there is no true magnetism, it is clearly immaterial whether the auxiliary surface ω follows the motion of the material parts or suffers a displacement independently of these, provided it remains within the space referred to, and is bounded by the curve s. In this case we may more simply and yet definitely assert that the integral of the electric force taken around the closed curve is equal to the time-rate of variation of the number of magnetic lines of force embraced by the curve s, multiplied by A. If we retain this same supposition, and if in addition the magnetic polarisation at every fixed point of the space is constant in spite of the motion of the curve s, we may assert that the induced force along the curve is equal to A multiplied by the number of magnetic lines of force, considered as at rest, which the curve s cuts in a given direction during its motion. If the magnetic forces, under the influence of which the curve s moves, are simply and solely due to the influence of an uniform current along a path t, then the number of lines of force traversing s is, as we have seen,[1] equal to the product of the Neumann's potential of the curves s and t, and of the current in t. In this case, therefore, the variation in the above-mentioned product per unit time multiplied by A gives the electromotive force acting along the curve s.

In one form or another these theorems embrace all known cases of induction which have been carefully investigated. The laws of unipolar induction, too, can be easily deduced from the general propositions. Quantitative investigations of induction-phenomena in bodies of three dimensions have only been carried out to a limited extent. The equations by which

[1] See p. 232.

Jochmann[1] and others have succeeded in representing the known facts, follow directly from our general equations by omitting a number of terms which naturally disappear in consequence of the special nature of the problem

We must not omit to mention that we may represent the general theorem of induction in another and a very elegant form if we allow ourselves to speak of an independent motion of the lines of force, and to regard in general every variation of the magnetic polarisation as the result of such a motion of the lines of force. If we do this, we may state generally and completely that the induced electromotive force in any closed curve s is equal to the product of A into the number of lines of magnetic force which are cut by the curve s in a definite sense per unit time. But although no objection can be raised to the occasional use of the conception therein involved, nevertheless it will be better for us to avoid it in the present paper. For the conception employed by Faraday, and developed by Poynting,[2] of a motion of the lines of force relatively to the surrounding medium, is indeed a highly remarkable one, and may be capable of being worked out; but it is entirely different from the view here followed, according to which the lines of force simply represent a symbol for special conditions of matter. There is no meaning in speaking of an independent motion of such conditions. It should also be observed that the controllable decrease and increase of the lines of force in all parts of the space does not definitely determine the presupposed motion of the lines of force. Hence the abovementioned proposition would not of itself decide definitely the magnitude of the induction in all cases; it should rather be regarded as a definition by means of which one among the possible motions of the lines of force is pointed out as the effective motion.

5. *Treatment of Surfaces of Slip*

At the boundary of two heterogeneous bodies the electromagnetic constants may pass from one value to another discontinuously; but the velocity-components $\alpha \beta \gamma$ do not

[1] Jochmann, *Crelle's Journ.* **63**, p. 1, 1863.

[2] J. H. Poynting, *Phil. Trans.* **2**, p. 277, 1885. [See also Note 35 at end of book.]

necessarily undergo discontinuous changes at the same time at this bounding surface. The surfaces of separation between solid bodies and fluids, or between fluids themselves, are to be regarded as surfaces of discontinuity of this kind; and we are free to suppose that the transition at the boundary between a body and the ether is of the same nature. The appearance of continuous motion at such surfaces of discontinuity does not give rise to any new considerations; the conditions of the material parts on both sides of the surface are connected by the same relations as those which obtain for bodies at rest.

But the case is different when the velocity-components also undergo discontinuous variations at the surface. As observed in our Introduction, the discontinuity can only refer to the components of the velocity which are parallel to the surface of separation; we therefore rightly denote surfaces of this kind as surfaces of slip (*Gleitflächen*). They may exist between solid bodies which are in contact with one another; it is also occasionally convenient and—seeing how ignorant we are as to the actual circumstances—permissible, to regard the surface of separation between a body and the ether as a surface of slip. As we have already remarked in the introduction, we treat a surface of slip as the limiting case of a transition-layer in which the motions, and possibly the electromagnetic constants as well, change very rapidly, but still continuously from one value to another. This conception is justified by the fact that it does not lead to any results in contradiction with experience; and it enables us to assert that the general propositions which we have already deduced do not become invalid in a system in which there are surfaces of slip. In order that our conception may suffice to determine the conditions in the surface of separation, the nature of the transition must be subjected to certain general restrictions. We give these restrictions in the form of hypotheses respecting the finiteness of certain magnitudes in the transition-layer itself. We assume that there are no electromotive forces at the surface of slip. We place the origin of the system of co-ordinates to which we refer at any point of the element of the transition-layer under consideration, and let it also follow this point during the motion. We further give the z-axis such a direction that it stands perpendicular to the element of the surface of slip, and also remains perpendicular during the motion. Thus the transition-layer always forms the immediate neighbourhood of the *xy*-plane. We assume that even in the transition-layer itself the quantities

$$\begin{array}{cccccc} X & Y & Z & L & M & N \\ \mathfrak{X} & \mathfrak{Y} & \mathfrak{Z} & \mathfrak{L} & \mathfrak{M} & \mathfrak{N} \\ u & v & w & \alpha & \beta & \gamma \end{array}$$

remain finite; and in the same way that the differential coefficients of these quantities parallel to the surface of slip, *i.e.* with respect to x and y, and also the differential coefficients of the quantities

$$\mathfrak{X} \quad \mathfrak{Y} \quad \mathfrak{Z} \quad \mathfrak{L} \quad \mathfrak{M} \quad \mathfrak{N}$$

with reference to the time t, remain finite. On the other hand, we should allow the differential coefficients with respect to z to become infinite, with the exception of $d\gamma/dz$, which, in accordance of the remark in the Introduction already referred to, must remain finite. Everywhere in the transition-layer, accordingly, γ itself is vanishingly small. These assumptions being made, we multiply the first two equations of the system (1_a) and (1_b) by dz, integrate with respect to z through the transition-layer between two points lying exceedingly near to it, and observe that, on account of the shortness of the integration-path, the integral of every quantity which remains finite in the layer vanishes. We thus obtain the following four equations, in which the index 1 refers to the one side, the index 2 to the other side, of the surface of separation—

$$(5_a) \quad A\int_1^2 \mathfrak{N}\frac{d\alpha}{dz}dz = Y_2 - Y_1, \quad -A\int_1^2 \mathfrak{N}\frac{d\beta}{dz}dz = X_2 - X_1;$$

$$(5_b) \quad -A\int_1^2 \mathfrak{Z}\frac{d\alpha}{dz}dz = M_2 - M_1, \quad A\int_1^2 \mathfrak{Z}\frac{d\beta}{dz}dz = L_2 - L_1.$$

These equations give the mutual relations between the force-components tangential to the surface of separation on both sides of it. Here, as in the case of bodies at rest, the components normal to the surface are connected by the condition that the surface-density of the true magnetism at the surface of separation cannot alter excepting by convection, and that the surface-density of the true electricity can only alter either by convection or by an actual current.

If the element of the surface of separation under consideration is not charged with any true electricity or true magnetism, $\mathfrak{Z}$ and $\mathfrak{N}$ are constant in the interior of the transition-layer. In this case the equations (5_a) and (5_b) take the simpler forms

$$(5_c)\quad X_2 - X_1 = A\mathfrak{N}(\beta_1 - \beta_2), \qquad Y_2 - Y_1 = A\mathfrak{N}(a_2 - a_1),$$
$$(5_d)\quad L_2 - L_1 = A\mathfrak{Z}(\beta_2 - \beta_1), \qquad M_2 - M_1 = A\mathfrak{Z}(a_1 - a_2).$$

As an example of the application of these equations, let us consider the case of a solid of revolution rotating about its axis within a hollow in another solid body which closely surrounds it. If this system is submitted to the action of a magnetic field which is symmetrical with reference to the axis of rotation, there will not be, according to our conception, either in the interior of the rotating body, or in the interior of the surrounding mass, any cause for the appearance of electric forces. Such forces are, in fact, absent when the magnetic excitement is entirely restricted to the interior of the one body or of the other. But if the lines of force penetrate through the surface along which the two bodies slide past one another, the electromotive forces expressed by equation (5_c) are excited at this surface; these forces spread into the interior of the bodies and there produce the electric stresses and currents whose existence is shown by experiment. If the bodies under consideration are non-conductors and are subjected to the influence of electric forces which are distributed symmetrically with reference to the axis of rotation, and which do not vanish at the surface of slip, the introduction of the motion excites magnetic forces in the neighbourhood in accordance with equation (5_d). It is true that effects of this kind cannot be observed with the same certainty as those first referred to; but there is at least an indication of them in Professor Röntgen's experiments.[1]

In the general case in which there are charges of true electricity and true magnetism at the surface of separation, a knowledge of the surface-density of these is not by itself sufficient for ascertaining the integrals of the equations (5_a)

[1] W. C. Röntgen, *Wied. Ann.* **35**, p. 264, 1888.

and (5_b); beyond this it is necessary to know to what extent the electricity and magnetism in the transition-layer share in the motion of each of the two contiguous bodies. This indeterminateness lies in the very nature of the matter. Consider, for example, Rowland's experiment on the effect of the convective motion of electricity; and suppose the electrified disc to rotate within a solid insulator surrounding it closely, instead of rotating in air. Clearly the magnetic effect would diminish, even to the point of vanishing entirely, as the electricity gradually escaped from the surface of the rotating disc on to the contiguous surface of the body at rest.

6. *Conservation of Energy—Ponderomotive Forces*

We shall consider the transition of the system from the initial to the final state during any element of time as being split up into two stages. In the first stage we shall suppose all the material parts to be transferred from their initial to their final position, the lines of force simply following the motion of the material parts. In the second stage we shall suppose that the electric and magnetic forces, which by this time are present, come into action, and in turn transfer the electromagnetic conditions into their final state. The variation which the electromagnetic energy of the system experiences during the whole period of transition is the sum of the variations which it experiences during the two stages. The processes which take place during the second stage are processes in bodies at rest; we already know how the variations of the electromagnetic energy during such processes are compensated by other forms of energy. But during the first stage, too, the electromagnetic energy of each material part of the system alters; we have therefore to account for what becomes of the electromagnetic energy thus diminished, or to find the source of any increase. As far as all existing experience extends, it can be proved beyond doubt that in every self-contained electromagnetic system the amount of energy in question is balanced by the mechanical work which is done by the electric and magnetic ponderomotive forces of the system during the element of time under consideration. But, nevertheless, taken as a statement of general applicability, this is not sufficient to enable us to deduce generally and rigidly the ponderomotive forces from the calculable variations of the electromagnetic energy. For this reason we introduce two further assumptions which are not inconsistent with it; these are not required by experience but by our own particular views. The first assumption declares that the statement already made—which experience proves to be correct for every self-contained electromagnetic system—also holds good for any material part of such a system. The second assumption asserts that no part of the system can exert upon the rest of the system any ponderomotive forces excepting pressures which are exerted by the elements of the first part upon the contiguous elements of the remaining part, and which at every point of the surface of contact depend simply upon the electromagnetic conditions of the immediate neighbourhood. The pressures required by the second assumption are determined without ambiguity by the first assumption; we shall deduce the magnitude of these pressures, and shall show that they are sufficient to explain the facts which have been directly observed. It then follows from the mode in which the pressures are deduced that the principle of the conservation of energy is satisfied in the case of moving bodies as well.

Consider during an element of time dt the magnetic energy of a material particle, whose varying volume may be denoted by $d\tau'$, while $d\tau$ denotes the value of $d\tau'$ at the beginning of the time-element dt. For the sake of simplicity let the origin of our system of co-ordinates be placed permanently in a material point of the space $d\tau'$. If $d\tau'$ moved as a rigid body, carrying its lines of force with it, the amount of energy contained in it would not alter. In general, therefore, the variation of this energy must be simply a function of the distortion which $d\tau'$ experiences in consequence of the motion; our immediate problem is to represent the variation in this form. Now it is not the polarisations alone which alter in consequence of the distortions, but also the properties of the material vehicles thereof, *i.e.* the magnetic constants. For the purpose of calculating this variation we need a further extension of our notation. In the first place, and in addition to the constants μ, we define a series of constants μ' by the condition that

$$\mathfrak{L}\mathrm{L} + \mathfrak{M}\mathrm{M} + \mathfrak{N}\mathrm{N}$$
$$= \mu_{11}\mathrm{L}_2 + 2\mu_{12}\mathrm{LM} + \text{etc.}$$
$$= \mu_{11}'\mathfrak{L}_2 + 2\mu_{12}'\mathfrak{L}\mathfrak{M} + \text{etc.}$$

These constants μ' are therefore the coefficients of $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ in the linear functions of these quantities by which the forces are represented. We further denote by $\xi\,\eta\,\zeta$ the displacements which the point, whose velocities are $\alpha\,\beta\,\gamma$, suffers from its original position at the beginning of the time dt. The quantities

$$\frac{d\xi}{dx} = x_x, \qquad \frac{d\xi}{dy} + \frac{d\eta}{dx} = x_y, \text{ etc.,}$$ [1]

are then the components of the distortions of the element $d\tau'$ in which the displacements $\xi\,\eta\,\zeta$ occur. The constants μ' are functions of these quantities; moreover, they depend upon the rotations ρ, σ, τ which the element experiences during the distortion. During the element of time dt both x_x, x_y, etc., and ρ, σ, τ remain vanishingly small; hence the dependence is linear and is known to us, provided we are given the differential coefficients of μ' with respect to ρ, σ, τ, x_x, x_y, etc. The differential coefficients with respect to ρ, σ, τ can be calculated from the instantaneous values of μ' itself. But this is not possible for the differential coefficients with respect to x_x, x_y, etc., and we must therefore assume that we are otherwise given the quantities

$$\frac{d\mu_{11}'}{dx_x} = \mu_{11}'{}_{,\,11}, \qquad \frac{d\mu_{11}'}{dx_y} = \mu_{11}'{}_{,\,12}, \text{ etc.,}$$
$$\frac{d\mu_{12}'}{dx_x} = \mu_{12}'{}_{,\,11}, \qquad \frac{d\mu_{12}'}{dx_y} = \mu_{12}'{}_{,\,12}, \text{ etc. etc.}$$

The 36 constants so defined clearly correspond to the magnetic properties of the particular substance which fills the space $d\tau'$ in its instantaneous state of deformation. For our purpose we cannot dispense with a single one of these constants; nor can we *à priori* deduce a single one of them from the magnetic properties of the substance which we have hitherto considered. By a suitable orientation of our system of co-

[1] Cf. G. Kirchhoff, *Mechanik*, p. 123, 1877.

ordinates we can reduce the number of necessary constants; similarly a reduction takes place when there happen to be symmetrical relations with respect to the system of co-ordinates used. In the simplest case, in which the substance is not only isotropic in its initial state, but also remains isotropic in spite of every deformation that arises—viz. in a fluid,—the number of the new constants reduces to a single one, which then, together with the one magnetic permeability, sufficiently defines the magnetic properties. Besides, it does not seem improbable that even in the general case necessary relations may be proved to exist between the constants which would then reduce to a smaller number of independent constants.

This notation being now assumed, we obtain successively the following expressions for the variation per unit time of the amount of magnetic energy contained in the space $d\tau'$:—

$$
(6)\begin{cases} \dfrac{d}{dt}\left\{\dfrac{1}{8\pi}(\mathfrak{L}L+\mathfrak{M}M+\mathfrak{N}N)d\tau'\right\} \\ \quad = \dfrac{1}{8\pi}\left\{d\tau\dfrac{d}{dt}(\mu_{11}'\mathfrak{L}^2+2\mu_{12}'\mathfrak{L}\mathfrak{M}+\text{etc.}) \right. \\ \qquad \left. + (\mathfrak{L}L+\mathfrak{M}M+\mathfrak{N}N)\dfrac{d}{dt}d\tau'\right\} \\ \quad = \dfrac{1}{8\pi}d\tau\left\{2\left(L\dfrac{d\mathfrak{L}}{dt}+M\dfrac{d\mathfrak{M}}{dt}+N\dfrac{d\mathfrak{N}}{dt}\right)\right. \\ \qquad + \left(\dfrac{d\mu_{11}'}{dt}\mathfrak{L}^2+2\dfrac{d\mu_{12}'}{dt}\mathfrak{L}\mathfrak{M}+\text{etc.}\right) \\ \qquad \left. + (\mathfrak{L}L+\mathfrak{M}M+\mathfrak{N}N)\left(\dfrac{d\alpha}{dx}+\dfrac{d\beta}{dy}+\dfrac{d\gamma}{dz}\right)\right\}. \end{cases}
$$

From the last of these we proceed to remove the differential coefficients with respect to t. We obtain the following expressions for $d\mathfrak{L}/dt$, $d\mathfrak{M}/dt$, $d\mathfrak{N}/dt$ from equations (1_a) by considering only the influence of the motion in them, and putting the velocities α, β, γ, with due regard to the special choice of our co-ordinate system, equal to zero—

$$
\begin{aligned}
\frac{d\mathfrak{L}}{dt} &= -\mathfrak{L}\left(\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right)+\mathfrak{M}\frac{d\alpha}{dy}+\mathfrak{N}\frac{d\alpha}{dz}, \\
\frac{d\mathfrak{M}}{dt} &= -\mathfrak{M}\left(\frac{d\gamma}{dz}+\frac{d\alpha}{dx}\right)+\mathfrak{N}\frac{d\beta}{dz}+\mathfrak{L}\frac{d\beta}{dx}, \\
\frac{d\mathfrak{N}}{dt} &= -\mathfrak{N}\left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}\right)+\mathfrak{L}\frac{d\gamma}{dx}+\mathfrak{M}\frac{d\gamma}{dy}.
\end{aligned}
$$

For the magnitude $d\mu_{11}'/dt$ we further have

$$\frac{d\mu_{11}'}{dt} = \frac{d\mu_{11}'}{dx_x} \cdot \frac{dx_x}{dt} + \frac{d\mu_{11}'}{dx_y} \cdot \frac{dx_y}{dt} + \text{etc.}$$

$$+ \frac{d\mu_{11}'}{d\rho} \cdot \frac{d\rho}{dt} + \text{etc.}$$

$$= \mu_{11}'{}_{11}\frac{da}{dx} + \mu_{11}'{}_{,12}\left(\frac{da}{dy} + \frac{d\beta}{dx}\right) + \text{etc.}$$

$$+ \tfrac{1}{2}\frac{d\mu_{11}'}{d\rho}\left(\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right) + \text{etc.}$$

We deduce similar expressions for $d\mu_{12}'/dt$, etc. We introduce all these expressions in the right-hand side of equation (6), and this side now becomes a homogeneous linear function of the nine differential coefficients of $a\,\beta\,\gamma$ with respect to $x\,y\,z$. But we can and will arrange this function so that it shall appear to us as a homogeneous linear function of the six rates of deformation da/dx, $da/dy + d\beta/dx$, etc., and of the three rates of rotation $\frac{1}{2}(da/dy - d\beta/dx)$, etc. We here note that the coefficients of the three rates of rotation must necessarily vanish identically; for a motion of a particle as a rigid body does not bring about any alteration in the amount of energy contained in it. Accordingly, we simply reject the terms in which these rates of rotation occur, and thus obtain as our final result, after reducing to the unit of volume by dividing by $d\tau$—

$$(6_a)\left\{\begin{aligned}
&\frac{1}{d\tau}\frac{d}{dt}\left\{\int\frac{1}{8\pi}(\mathfrak{L}L + \mathfrak{M}M + \mathfrak{N}N)d\tau'\right\}\\
&= \frac{1}{8\pi}\frac{da}{dx}(\ \ \mathfrak{L}L - \mathfrak{M}M - \mathfrak{N}N + \mu_{11}'{}_{,11}\mathfrak{L}^2 + 2\mu_{12}'{}_{,11}\mathfrak{L}\mathfrak{M} + \text{etc.})\\
&+ \frac{1}{8\pi}\frac{d\beta}{dy}(-\mathfrak{L}L + \mathfrak{M}M - \mathfrak{N}N + \mu_{11}'{}_{,22}\mathfrak{L}^2 + 2\mu_{12}'{}_{,22}\mathfrak{L}\mathfrak{M} + \text{etc.})\\
&+ \frac{1}{8\pi}\frac{d\gamma}{dz}(-\mathfrak{L}L - \mathfrak{M}M + \mathfrak{N}N + \mu_{11}'{}_{,33}\mathfrak{L}^2 + 2\mu_{12}'{}_{,33}\mathfrak{L}\mathfrak{M} + \text{etc.})\\
&+ \frac{1}{8\pi}\left(\frac{d\beta}{dz} + \frac{d\gamma}{dy}\right)(\ \ \mathfrak{N}M + \mathfrak{M}N + \mu_{11}'{}_{,23}\mathfrak{L}^2 + 2\mu_{12}'{}_{,23}\mathfrak{L}\mathfrak{M} + \text{etc.})\\
&+ \frac{1}{8\pi}\left(\frac{d\gamma}{dx} + \frac{da}{dz}\right)(\ \ \mathfrak{L}N + \mathfrak{N}L + \mu_{11}'{}_{,13}\mathfrak{L}^2 + 2\mu_{12}'{}_{,13}\mathfrak{L}\mathfrak{M} + \text{etc.})\\
&+ \frac{1}{8\pi}\left(\frac{da}{dy} + \frac{d\beta}{dx}\right)(\ \ \mathfrak{M}L + \mathfrak{L}M + \mu_{11}'{}_{,12}\mathfrak{L}^2 + 2\mu_{12}'{}_{,12}\mathfrak{L}\mathfrak{M} + \text{etc.})
\end{aligned}\right.$$

Now it is clear that in the linear function of the rates of distortion on the right hand the coefficient, taken negatively, of each of these rates, gives that pressure-component with which the magnetically strained matter tends to increase the corresponding distortion. For let us, in accordance with the usual[1] notation, denote by $X_x X_y X_z$ the components of the pressure which the matter of the element $d\tau$ exerts upon a plane section perpendicular to the x-axis; and let us further extend this notation to the directions of the other axes. Then the expression

$$X_x\frac{d\alpha}{dx}+Y_y\frac{d\beta}{dy}+Z_z\frac{d\gamma}{dz}$$
$$+Y_z\left(\frac{d\beta}{dz}+\frac{d\gamma}{dy}\right)+X_z\left(\frac{d\gamma}{dx}+\frac{d\alpha}{dz}\right)+X_y\left(\frac{d\alpha}{dy}+\frac{d\beta}{dx}\right)$$

represents the mechanical work, per unit volume and per unit time, done by the material contents of the element $d\tau$ in the distortion which takes place. According to our assumption this mechanical work is equal to the magnetic energy which is lost as a result of the distortion. Inasmuch as this holds for every possible deformation our assertion is shown to be correct. Each of the pressure-components obtained is a homogeneous quadratic function of the three components of the prevailing magnetic force or, similarly, of the three components of the prevailing magnetic polarisation. By exactly analogous considerations we can deduce exactly analogous expressions for the pressures which arise through electric stresses. The total pressure is equal to the sum of the magnetic and electric pressures.

Having now found the values of the ponderomotive pressures, we add three remarks. The first remark has reference to the difference between our system of pressures and the system which Maxwell has given for the general case in which the forces and the polarisations have different directions.[2] In the first place, Maxwell's formulæ are simpler, because in deducing them he paid no heed to the possible deformation of the medium. A much more important difference consists in the fact that the force-components which, according to the notation

[1] G. Kirchhoff (*Mechanik*, Eleventh Lecture).
[2] Maxwell, *Treatise on Elect. and Mag.*, 2, p. 254, 1873.

used, are denoted by X_y and Y_x, have different values in Maxwell, whereas with us they are identical. According to our system each material particle, when left to itself, simply changes its form; according to Maxwell's system it would at the same time begin to rotate as a whole. Hence Maxwell's pressures cannot owe their origin to processes in the interior of the element; and they therefore find no place in the theory here worked out. At the same time they are permissible, if one starts with the assumption that in the interior of the moving body the ether remains permanently at rest and provides the necessary point of support for the rotation which takes place.

The second remark has reference to the manner in which our formulæ become simplified when we apply them to bodies which are isotropic, and which, in spite of every deformation, remain isotropic—viz. to fluids. The system of constants μ' here reduces to the one constant $\mu' = 1/\mu$. If we further denote by σ the density of the fluid, we have

$$\mu'_{11,\,11} = \mu'_{22,\,22} = \mu'_{33,\,33} = -\frac{d\left(\frac{1}{\mu}\right)}{d\log\sigma} = \frac{1}{\mu^2}\frac{d\mu}{d\log\sigma},$$

$$\mu'_{12,\,11} = \text{etc.} = 0.$$

Thus the pressure-components are—

$$(6_b)\begin{cases} X_x = \dfrac{\mu}{8\pi}(-L^2 + M^2 + N^2) - \dfrac{d\mu}{8\pi d\log\sigma}(L^2 + M^2 + N^2), \\ Y_y = \dfrac{\mu}{8\pi}(L^2 - M^2 + N^2) - \dfrac{d\mu}{8\pi d\log\sigma}(L^2 + M^2 + N^2), \\ Z_z = \dfrac{\mu}{8\pi}(L^2 + M^2 - N^2) - \dfrac{d\mu}{8\pi d\log\sigma}(L^2 + M^2 + N^2), \end{cases}$$

$$X_y = -\frac{\mu}{4\pi}LM, \qquad X_z = -\frac{\mu}{4\pi}NL, \qquad Y_z = -\frac{\mu}{4\pi}MN.$$

For the same case quite identical formulæ have already been obtained by von Helmholtz[1] by following a similar train of thought. Our formulæ merge into his if we alter the notation so as to replace L, M, N and μ by λ/ϑ, μ/ϑ, ν/ϑ, and

[1] v. Helmholtz, *Wied. Ann.*, **13**, p. 400, 1881.