# XIII

# ON THE FUNDAMENTAL EQUATIONS OF ELECTROMAGNETICS FOR BODIES AT REST

(*Göttinger Nachr.* March 19, 1890; *Wiedemann's Ann.* **40**, p. 577)

THE system of ideas and formulæ by which Maxwell represented electromagnetic phenomena is in its possible developments richer and more comprehensive than any other of the systems which have been devised for the same purpose. It is certainly desirable that a system which is so perfect, as far as its contents are concerned, should also be perfected as far as possible in regard to its form. The system should be so constructed as to allow its logical foundations to be easily recognised; all unessential ideas should be removed from it, and the relations of the essential ideas should be reduced to their simplest form. In this respect Maxwell's own representation does not indicate the highest attainable goal; it frequently wavers between the conceptions which Maxwell found in existence, and those at which he arrived. Maxwell starts with the assumption of direct actions-at-a-distance; he investigates the laws according to which hypothetical polarisations of the dielectric ether vary under the influence of such distance-forces; and he ends by asserting that these polarisations do really vary thus, but without being actually caused to do so by distance-forces.[1] This procedure leaves behind it the unsatisfactory feeling that there must be something wrong about either the final result or the way which led to it. Another effect of this procedure is that in the formulæ there are retained a number of superfluous,

[1] The same remark applies to v. Helmholtz's paper in vol. **72** of Crelle's *Journal*,—not, indeed, throughout, but as far as relates to the special values of the constants, which allow the distance-forces to vanish from the final results, and which, therefore, lead to the theory here supported.

and in a sense rudimentary, ideas which only possessed their proper significance in the older theory of direct action-at-a-distance. Among such rudimentary ideas of a physical nature I may mention that of dielectric displacement in free ether, as distinguished from the electric force which produces it and the relation between the two—the specific inductive capacity of the ether. These distinctions have a meaning so long as we can remove the ether from a space and yet allow the force to persist in it. This was conceivable, according to the conception from which Maxwell started; it is not conceivable, according to the conception to which we have been led by his researches. As a rudimentary idea of a mathematical nature I may mention the predominance of the vector-potential in the fundamental equations. In the construction of the new theory the potential served as a scaffolding; by its introduction the distance-forces which appeared discontinuously at particular points were replaced by magnitudes which at every point in space were determined only by the conditions at the neighbouring points. But after we have learnt to regard the forces themselves as magnitudes of the latter kind, there is no object in replacing them by potentials unless a mathematical advantage is thereby gained. And it does not appear to me that any such advantage is attained by the introduction of the vector-potential in the fundamental equations; furthermore, one would expect to find in these equations relations between the physical magnitudes which are actually observed, and not between magnitudes which serve for calculation only.

Again, the incompleteness of form referred to renders it more difficult to apply Maxwell's theory to special cases. In connection with such applications I have been led to endeavour for some time past to sift Maxwell's formulæ and to separate their essential significance from the particular form in which they first happened to appear. The results at which I have arrived are set forth in the present paper. Mr. Oliver Heaviside has been working in the same direction ever since 1885. From Maxwell's equations he removes the same symbols as myself; and the simplest form which these equations[1]

[1] These equations will be found in the *Phil. Mag.* for February 1888. Reference is there made to earlier papers in the *Electrician* for 1885, but this source was not accessible to me.

thereby attain is essentially the same as that at which I arrive. In this respect, then, Mr. Heaviside has the priority. Nevertheless, I hope that the following representation will not be deemed superfluous. It does not claim to set forth matters in a final form; but only in such a manner as to admit of further improvements more easily than has hitherto been possible.

I divide the subject into two parts. In the first part (A) I give the fundamental ideas and the formulæ by which they are connected. Explanations will be added to the formulæ; but these explanations are not to be regarded as proofs of the formulæ. The statements will rather be given as facts derived from experience; and experience must be regarded as their proof. It is true, meanwhile, that each separate formula cannot be specially tested by experience, but only the system as a whole. But practically the same holds good for the system of equations of ordinary dynamics.

In the second part (B) I state in what manner the facts which are directly observed can be systematically deduced from the formulæ; and, hence, by what experiences the correctness of the system can be proved. A complete treatment of this part would naturally assume very large dimensions, and therefore mere indications must here suffice.

## A. The Fundamental Ideas and their Connection

### 1. *Electric and Magnetic Force*

Starting from rest, the interior of all bodies, including the free ether, can experience disturbances which we denote as electrical, and others which we denote as magnetic. The nature of these changes of state we do not know, but only the phenomena which their presence causes. Regarding these latter as known we can, with their aid, determine the geometrical relations of the changes of state themselves. The disturbances of the electric and the magnetic kind are so connected with one another that disturbances of the one kind can continuously exist independently of those of the other kind; but that, on the other hand, it is not possible for disturbances of either of the two kinds to experience temporary fluctuations without exciting simultaneously disturbances of the other kind.

The production of the change of state necessitates an expenditure of energy; this energy is again released when the disturbance disappears; hence the presence of the disturbance represents a stock of energy. At any given point the changes of state of either kind can be distinguished as to direction, sense, and magnitude. For the determination, therefore, of the electrical as well as of the magnetic state, it is necessary to specify a directed magnitude or the three components thereof. But it is an essential and important hypothesis of our present theory that the specification of a single directed magnitude is sufficient to determine completely the change of state under consideration. Certain phenomena, *e.g.* those of permanent magnetism, dispersion, etc., are not intelligible from this standpoint; they require that the electric or magnetic conditions of any point should be represented by more than one variable.[1] For this very reason such phenomena are excluded from our considerations in their present state.

That directed magnitude by means of which we first determine the electrical state, we call the electric force. The phenomenon by which we define the electric force is the mechanical force which a certain electrified body experiences in empty space under electrical stress. That is to say, for empty space we make the component of the electric force in any given direction proportional to the component of this mechanical force in the same direction. By electric force at a point in a ponderable body we understand the electric force at this point inside an infinitely small cylindrical space, infinitely narrow as compared with its length, bored out of the body in such a way that its direction coincides with that of the force—a requirement which, as experience shows, can always be satisfied. And whatever may be the relation between the force so measured and the actual change of state of the body, it certainly must, in accordance with our hypothesis, determine the change of state without ambiguity. If we everywhere replace the word "electric" by the word "magnetic," and the electrified test-body by the pole of a magnetic needle, we obtain the definition of magnetic force. In order to settle the sense of both forces in the conventional manner, let us further stipulate that the electrified test-body is charged with vitreous electricity, and

[1] [See Note 29 at end of book.]

that the pole of the magnetic needle used is the one which points towards the north. The units of the forces are still reserved. The components of the electric force in the directions $x$, $y$, $z$, we shall denote as X, Y, Z, and the corresponding components of the magnetic force as L, M, N.

## 2. *The Energy of the Field*

The stock of electrical energy in a portion of a body, within which the electric force has a constant value, is a homogeneous quadratic function of the three components of the electric force. The corresponding statement holds good for the supply of magnetic energy. The total supply of energy we shall denote as the electromagnetic; it is the sum of the electrical and the magnetic.

According to this, the amount of energy of either kind per unit volume is for an isotropic body equal to the product of the square of the total force under consideration and a constant. The magnitude of the latter may be different for the electric and the magnetic energy; it depends upon the material of the body and the choice of the units for energy and for the forces. We shall measure the energy in absolute Gauss's measure; and shall now fix the units of the forces by stipulating that in free ether the value of the constants shall be equal to $1/8\pi$, so that the energy of unit volume of the stressed ether will be

$$\frac{1}{8\pi}(X^2+Y^2+Z^2)+\frac{1}{8\pi}(L^2+M^2+N^2).$$

When the forces are measured in this manner, we say that they are measured in absolute Gauss's units.[1] The dimensions of the electric force become the same as those of the magnetic force. Both are such that their square has the dimensions of energy per unit volume; or, expressed in the usual notation, the dimensions of both are $M^{1/2}L^{-1/2}T^{-1}$.

For every isotropic ponderable body we can now, in accordance with what has been stated, put the energy per unit volume as equal to

$$\frac{\epsilon}{8\pi}(X^2+Y^2+Z^2)+\frac{\mu}{8\pi}(L^2+M^2+N^2).$$

[1] See H. Helmholtz, *Wied. Ann.* **17**, p. 42, 1882.

The new constants ($\epsilon$ and $\mu$) here introduced are necessarily positive, and are simply numbers. We shall call $\epsilon$ the specific inductive capacity (*Dielektricitätsconstante*) and $\mu$ the magnetic permeability (*Magnetisirungsconstante*) of the substance. Clearly $\epsilon$ and $\mu$ are numerical ratios, by means of which we compare the energy of one material with the energy of another material. A definite value of either does not follow simply from the nature of a single substance itself. This is what we mean when we say that the specific inductive capacity and the magnetic permeability are not intrinsic constants of a substance. There is nothing wrong in saying that these constants are equal to unity for the ether; but this does not state any fact derived from experience; it is only an arbitrary stipulation on our part.

For crystalline bodies the energy per unit of volume will be equal to

$$\frac{1}{8\pi}(\epsilon_{11}X^2 + \epsilon_{22}Y^2 + \epsilon_{33}Z^2 + 2\epsilon_{12}XY + 2\epsilon_{23}YZ + 2\epsilon_{13}XZ)$$
$$+ \frac{1}{8\pi}(\mu_{11}L^2 + \mu_{22}M^2 + \mu_{33}N^2 + 2\mu_{12}LM + 2\mu_{23}MN + 2\mu_{13}LN).$$

By a suitable choice of axes either the one part or the other of this expression can be transformed into a sum of three squares. It is even probable that the same choice of axes would thus transform both parts. The $\epsilon$ and $\mu$ must be such that in the transformation into a sum of squares all the coefficients would become positive.

### 3. *Connection of the Forces in the Ether*

We assume that the system of co-ordinates is such that the direction of positive $x$ is straight out in front of us, the direction of positive $z$ upwards, and that $y$ increases from left to right.[1] Assuming this, the electric and magnetic forces in the ether are connected with each other according to the following equations:—

[1] Unfortunately for the English reader this is not the system employed by Maxwell, but the symmetric one. Hence follow some differences from Maxwell's formulæ as to the signs + and −. The system is that which is employed in v. Helmholtz's papers.

$$
(3_a)\begin{cases} A\dfrac{dL}{dt} = \dfrac{dZ}{dy} - \dfrac{dY}{dz}, \\ A\dfrac{dM}{dt} = \dfrac{dX}{dz} - \dfrac{dZ}{dx}, \\ A\dfrac{dN}{dt} = \dfrac{dY}{dx} - \dfrac{dX}{dy}; \end{cases} \qquad (3_b)\begin{cases} A\dfrac{dX}{dt} = \dfrac{dM}{dz} - \dfrac{dN}{dy}, \\ A\dfrac{dY}{dt} = \dfrac{dN}{dx} - \dfrac{dL}{dz}, \\ A\dfrac{dZ}{dt} = \dfrac{dL}{dy} - \dfrac{dM}{dx}, \end{cases}
$$

in addition to which we have the equations (which are not inconsistent with the above)—

$$
(3_c) \quad \frac{dL}{dx} + \frac{dM}{dy} + \frac{dN}{dx} = 0, \qquad \frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 0
$$

as a supplement distinguishing the ether from ponderable matter.

After these equations are once found, it no longer appears expedient to deduce them (in accordance with the historical course) from conjectures as to the electric and magnetic constitution of the ether and the nature of the acting forces,—all these things being entirely unknown. Rather is it expedient to start from these equations in search of such further conjectures respecting the constitution of the ether.

Since the dimensions of X, Y, Z, and of L, M, N are the same, the constant A must be the reciprocal of a velocity. It is in reality an intrinsic constant of the ether; in saying this we assert that its magnitude is independent of the presence of any other body, or of any arbitrary stipulations on our part.

We multiply all our equations by $(1/4\pi A) . d\tau$; further multiply the members of the series separately by L, M, N, X, Y, Z respectively, and add all together. We integrate both sides of the resulting equation over any definite space, of which the element of surface $d\omega$ makes, with the co-ordinate axes, the angle $n,x$, $n,y$, $n,z$. The integration can be carried out on the right-hand side of the equation, and we get—

$$
\frac{d}{dt}\int \left\{ \frac{1}{8\pi}(X^2 + Y^2 + Z^2) + \frac{1}{8\pi}(L^2 + M^2 + N^2) \right\} d\tau
$$

$$
= \frac{1}{4\pi A}\int \{ (NY - MZ) \cos n,x + (LZ - NX) \cos n,y + (MX - LY) \cos n,z \} d\omega.
$$

The integral on the left-hand side is the electromagnetic energy of the space; the equation, therefore, gives us the variation of this energy, expressed in magnitudes which relate only to the bounding surface of the space.

### 4. *Isotropic Non-Conductors*

In homogeneous isotropic non-conductors the phenomena are qualitatively identical with those in free ether. Quantitatively they differ in two respects: in the first place, the intrinsic constant has a value different from what it has in the ether; and in the second place, the expression for the energy per unit volume contains, as already explained, the constants $\epsilon$ and $\mu$. The following equations represent these statements, and are in accord with experience:—

$$(4_a)\begin{cases} A\mu\dfrac{dL}{dt}=\dfrac{dZ}{dy}-\dfrac{dY}{dz}, \\ A\mu\dfrac{dM}{dt}=\dfrac{dX}{dz}-\dfrac{dZ}{dx}, \\ A\mu\dfrac{dN}{dt}=\dfrac{dY}{dx}-\dfrac{dX}{dy}; \end{cases} \qquad (4_b)\begin{cases} A\epsilon\dfrac{dX}{dt}=\dfrac{dM}{dz}-\dfrac{dN}{dy}, \\ A\epsilon\dfrac{dY}{dt}=\dfrac{dN}{dx}-\dfrac{dL}{dz}, \\ A\epsilon\dfrac{dZ}{dt}=\dfrac{dL}{dy}-\dfrac{dM}{dx}. \end{cases}$$

For if, for a moment, we determine the measure of the forces in the non-conductor as we have previously done in the ether, and accordingly replace X, Y, Z by $X/\sqrt{\epsilon}$, $Y/\sqrt{\epsilon}$, $Z/\sqrt{\epsilon}$, and L, M, N by $L/\sqrt{\mu}$, $M/\sqrt{\mu}$, $N/\sqrt{\mu}$; then the equations assume exactly the form of the equations for the ether—with this single difference, that the magnitude A is replaced by the magnitude $A/\sqrt{\epsilon\mu}$. If we retain, on the other side, our measure of the forces, we can consistently assign to the energy the requisite value. For by carrying out the same operations which we employed in the preceding section, we get—

$$\frac{d}{dt}\int\left\{\frac{\epsilon}{8\pi}(X^2+Y^2+Z^2)+\frac{\mu}{8\pi}(L^2+M^2+N^2)\right\}d\tau$$
$$=\frac{1}{4\pi A}\int\{(NY-MZ)\cos n,x+(LZ-NX)\cos n,y$$
$$+(MX-LY)\cos n,z\}d\omega.$$

The general statements by which we have been guided to equations (4) no longer hold good when the non-conductor is not homogeneous. The question therefore arises—Do our equations hold good in this case? Experience answers this question in the affirmative; we may therefore regard the magnitudes $\epsilon$ and $\mu$ in equations ($4_a$) and ($4_b$) as variable from point to point.

### 5. *Crystalline Non-Conductors*

We can obtain a representation of the processes that take place in such bodies—whose structure differs in different directions, but whose electromagnetic properties merge into those of isotropic non-conductors as the eolotropy disappears —by regarding the time-variations of the forces on the left hand of our equations as perfectly general linear functions of the space-variations of the forces of the opposite kind on the right hand. The generality of form of these linear functions and the choice of their constants is, however, restricted by assuming that the same operation which has already furnished us with an equation for the variation of energy will always do so, and by stipulating that the energy shall thereby be expressed in the form already established. By these considerations we are led to the following equations, which, in fact, suffice for the representation of the most important phenomena:—

$$(5_a)\left\{\begin{aligned} A\left(\mu_{11}\frac{dL}{dt}+\mu_{12}\frac{dM}{dt}+\mu_{13}\frac{dN}{dt}\right)&=\frac{dZ}{dy}-\frac{dY}{dz},\\ A\left(\mu_{12}\frac{dL}{dt}+\mu_{22}\frac{dM}{dt}+\mu_{23}\frac{dN}{dt}\right)&=\frac{dX}{dz}-\frac{dZ}{dx},\\ A\left(\mu_{13}\frac{dL}{dt}+\mu_{23}\frac{dM}{dt}+\mu_{33}\frac{dN}{dt}\right)&=\frac{dY}{dx}-\frac{dX}{dy};\end{aligned}\right.$$

$$(5_b)\left\{\begin{aligned} A\left(\epsilon_{11}\frac{dX}{dt}+\epsilon_{12}\frac{dY}{dt}+\epsilon_{13}\frac{dZ}{dt}\right)&=\frac{dM}{dz}-\frac{dN}{dy},\\ A\left(\epsilon_{12}\frac{dX}{dt}+\epsilon_{22}\frac{dY}{dt}+\epsilon_{23}\frac{dZ}{dt}\right)&=\frac{dN}{dx}-\frac{dL}{dz},\\ A\left(\epsilon_{13}\frac{dX}{dt}+\epsilon_{23}\frac{dY}{dt}+\epsilon_{33}\frac{dZ}{dt}\right)&=\frac{dL}{dy}-\frac{dM}{dx}.\end{aligned}\right.$$

The equation for the variation of the energy of a space gives the same result as in sections (3) and (4). Experience also shows that it is not necessary to regard the $\epsilon$ and $\mu$ in the equations of the present section as being constant throughout the space; they may be magnitudes varying in any way from point to point.

### 6. *Distribution of the Forces in Conductors*

In the bodies hitherto considered, every variation of the electric force appears as the consequence of the presence of magnetic forces. If within a finite region the magnetic forces are equal to zero, every cause for such a variation is wanting; and any existing distribution of electric force remains permanently, so long as it is left to itself and no disturbance reaches the interior from beyond the limits of the region. The electric forces do not behave thus in all bodies. In many bodies an electric force when left to itself vanishes more or less rapidly away; in such bodies magnetic forces or other causes are necessary in order to restrain an existing distribution from change. For reasons which will appear later, we call such bodies conductors. The simplest assumptions which we can make with respect to them are these: In the first place, that the loss per unit time experienced by an electric force when left to itself is proportional to the force itself; and, in the second place, that independently of this loss the magnetic forces here tend to produce the same variations as in the bodies hitherto considered. If we introduce a new constant $\lambda$, the first assumption allows us to assert that the force-component X when left to itself will vary in accordance with the equation—

$$A\epsilon\frac{dX}{dt} = -4\pi\lambda AX.$$

This first assumption is supplemented by the second as follows:—When magnetic forces are present, the variation will take place in accordance with the equation—

$$A\epsilon\frac{dX}{dt} = \frac{dM}{dz} - \frac{dN}{dy} - 4\pi\lambda AX.$$

The constant $\lambda$ is called the specific conductivity of the body, measured electrostatically. Its dimension is the reciprocal of a time. Hence the magnitude $\epsilon/4\pi\lambda$ is a time; it is the time in which the force when left to itself sinks to $1/e$ of its initial value—the so-called time of relaxation. Hr. E. Cohn first observed and drew attention to the fact[1] that it is this latter time, and not $\lambda$ itself, that is a second intrinsic constant of the body under consideration; one that can be settled without ambiguity and independently of any second medium.

These considerations lead us now, conjecturally, to the following equations which are in accordance with experience:—

$$(6_a)\begin{cases} A\mu\dfrac{dL}{dt}=\dfrac{dZ}{dy}-\dfrac{dY}{dz}, \\ A\mu\dfrac{dM}{dt}=\dfrac{dX}{dz}-\dfrac{dZ}{dx}, \\ A\mu\dfrac{dN}{dt}=\dfrac{dY}{dx}-\dfrac{dX}{dy}; \end{cases} \qquad (6_b)\begin{cases} A\epsilon\dfrac{dX}{dt}=\dfrac{dM}{dz}-\dfrac{dN}{dy}-4\pi\lambda AX, \\ A\epsilon\dfrac{dY}{dt}=\dfrac{dN}{dL}-\dfrac{dL}{dz}-4\pi\lambda AY, \\ A\epsilon\dfrac{dZ}{dt}=\dfrac{dx}{dy}-\dfrac{dM}{dx}-4\pi\lambda AZ. \end{cases}$$

Clearly these equations refer only to isotropic media; it is, however, unnecessary, as far as the hypotheses hitherto made are concerned, that the bodies should be homogeneous as well. But in order to represent accurately the actual distribution of the forces in a non-homogeneous body, our equations still need to be supplemented to a certain extent. For if the constitution of a body varies from point to point, the electric force when left to itself does not in general sink completely to zero, but it assumes a certain final value which is not zero. This value, whose components may be X′ Y′ Z′, we call the electromotive force acting at the point in question. We regard this as being independent of time; in general it is greater, the greater the variation of the chemical nature of the body per unit of length. We take into account the action of the electromotive force as follows:—Instead of making the diminution of the electric force when left to itself proportional to its absolute value, we make it proportional to the difference which remains between this absolute value and the final value. In the case, then, of conductors whose structure admits of the production of electromotive forces, our equations become—

[1] With respect to this, and the manner in which the magnitude $\lambda$ is here introduced, cf. E. Cohn, *Berl. Ber.* **26**, p. 405, 1889.

$$
(6_c)\begin{cases}
A\mu\dfrac{dL}{dt}=\dfrac{dZ}{dy}-\dfrac{dY}{dz}, \\
A\mu\dfrac{dM}{dt}=\dfrac{dX}{dz}-\dfrac{dZ}{dx}, \\
A\mu\dfrac{dN}{dt}=\dfrac{dY}{dx}-\dfrac{dX}{dy};
\end{cases}
\qquad
(6_d)\begin{cases}
A\epsilon\dfrac{dX}{dt}=\dfrac{dM}{dz}-\dfrac{dN}{dy}-4\pi\lambda A(X-X'), \\
A\epsilon\dfrac{dY}{dt}=\dfrac{dN}{dx}-\dfrac{dL}{dz}-4\pi\lambda A(Y-Y'), \\
A\epsilon\dfrac{dZ}{dt}=\dfrac{dL}{dy}-\dfrac{dM}{dx}-4\pi\lambda A(Z-Z').
\end{cases}
$$

## 7. *Eolotropic Conductors*

If the conductor behaves differently in different directions, we can no longer assume that the diminution in each component of the force when left to itself depends only upon this same component; we must rather suppose that it is a linear function of the three components. If we further assume that when the conductivity vanishes, the equations reduce to those of an eolotropic non-conductor, we arrive at the following system :—

$$
(7_a)\begin{cases}
A\left(\mu_{11}\dfrac{dL}{dt}+\mu_{12}\dfrac{dM}{dt}+\mu_{13}\dfrac{dN}{dt}\right)=\dfrac{dZ}{dy}-\dfrac{dY}{dz}, \\
A\left(\mu_{12}\dfrac{dL}{dt}+\mu_{22}\dfrac{dM}{dt}+\mu_{23}\dfrac{dN}{dt}\right)=\dfrac{dX}{dz}-\dfrac{dX}{dx}, \\
A\left(\mu_{13}\dfrac{dL}{dt}+\mu_{23}\dfrac{dM}{dt}+\mu_{33}\dfrac{dN}{dt}\right)=\dfrac{dY}{dx}-\dfrac{dX}{dy};
\end{cases}
$$

$$
(7_b)\begin{cases}
A\left(\epsilon_{11}\dfrac{dX}{dt}+\epsilon_{12}\dfrac{dY}{dt}+\epsilon_{13}\dfrac{dZ}{dt}\right)=\dfrac{dM}{dz}-\dfrac{dN}{dy} \\
\qquad -4\pi A\{\lambda_{11}(X-X')+\lambda_{12}(Y-Y')+\lambda_{13}(Z-Z')\}, \\
A\left(\epsilon_{12}\dfrac{dX}{dt}+\epsilon_{22}\dfrac{dY}{dt}+\epsilon_{23}\dfrac{dZ}{dt}\right)=\dfrac{dN}{dx}-\dfrac{dL}{dz} \\
\qquad -4\pi A\{\lambda_{21}(X-X')+\lambda_{22}(Y-Y')+\lambda_{23}(Z-Z')\}, \\
A\left(\epsilon_{13}\dfrac{dX}{dt}+\epsilon_{23}\dfrac{dY}{dt}+\epsilon_{33}\dfrac{dZ}{dt}\right)=\dfrac{dL}{dy}-\dfrac{dM}{dx} \\
\qquad -4\pi A\{\lambda_{31}(X-X')+\lambda_{32}(Y-Y')+\lambda_{33}(Z-Z')\}.
\end{cases}
$$

It is highly probable that for all actual bodies $\lambda_{12}=\lambda_{21}$, $\lambda_{31}=\lambda_{13}$, $\lambda_{23}=\lambda_{32}$. We may regard the constants $\epsilon$, $\mu$, $\lambda$ in the equations of this section again as varying in value from place to place.

## 8. *Limiting Conditions*

It is easily seen that the equations ($7_a$) and ($7_b$) include all the earlier ones as particular cases; and that even the equations for the free ether can be deduced from them by a suitable disposition of the constants. Now since these constants may be functions of the space, we are led to regard the surface of separation of two heterogeneous bodies as a transition-layer, in which the constants certainly pass with extraordinary rapidity from one value to another, but in which this still happens in such a way that even in the layer itself the above equations always hold good, and express finite relations between the finite values of the constants and the forces which also remain finite. In order to deduce the limiting conditions from this hypothesis, which is in accordance with experience, let us for the sake of simplicity suppose that the element of the surface of separation under consideration coincides with the $xy$-plane.

Disregarding for the moment the appearance of electromotive forces between the bodies in contact, we find, on examining the first two of the equations ($7_a$) and ($7_b$), that the magnitudes

$$\frac{dX}{dz}, \quad \frac{dY}{dz}, \quad \frac{dM}{dz}, \quad \frac{dL}{dz}$$

must, in consequence of our hypothesis, remain finite in the transition-layer as well. Thus, if the index 1 refers to the one side of the limiting layer, and the index 2 to the other side, we must have

$$(8_a)\quad \begin{aligned} Y_2 - Y_1 &= 0, \\ X_2 - X_1 &= 0. \end{aligned} \qquad\qquad (8_b)\quad \begin{aligned} M_2 - M_1 &= 0, \\ L_2 - L_1 &= 0. \end{aligned}$$

The components of the force which are tangential to the limiting surface therefore continue through it without discontinuity. Applying this to the third of the equations ($7_a$) and ($7_b$), we further find that the expressions

$$\mu_{13}\frac{dL}{dt} + \mu_{23}\frac{dM}{dt} + \mu_{33}\frac{dN}{dt} \quad \text{and}$$

$$\epsilon_{13}\frac{dX}{dt} + \epsilon_{23}\frac{dY}{dt} + \epsilon_{33}\frac{dZ}{dt} + 4\pi(\lambda_{31}X + \lambda_{32}Y + \lambda_{33}Z)$$

must have the same value on both sides of the limiting layer. This statement, which expresses the reciprocal dependence of the normal components of the forces on both sides of the limiting surface, assumes in the case of isotropic bodies the simple form

$$(8_c)\quad \mu_1 \frac{dN_1}{dt} - \mu_2 \frac{dN_2}{dt} = 0,$$

$$(8_d)\quad \epsilon_1 \frac{dZ_1}{dt} - \epsilon_2 \frac{dZ_2}{dt} = -4\pi(\lambda_1 Z_1 - \lambda_2 Z_2).$$

In the next place, if we do not exclude the appearance of electromotive forces in the limiting layer, we have to observe that, in accordance with experience, the component of these forces which is normal to the limiting surface, *i.e.* $Z'$, becomes infinite in the transition-layer itself; and yet in such a way that the integral $\int Z'dz$ taken through the limiting surface retains a finite value; this value we can obtain from experiments, although these leave us in the dark as to the course of $Z'$ itself. We now satisfy the hypothesis of the present section by assuming that, with L, M, N, X, Y, the magnitude $Z-Z'$ remains finite in the transition-layer. Z becomes infinite there; nevertheless, we can allow $dZ/dt$ to remain finite. Furthermore, we put

$$(8_e)\quad \int Zdz = \int Z'dz = \phi_{1,2},$$

Let us now integrate the first two of the equations $(7_a)$ and $(7_b)$ after multiplying by $dz$ through the transition-layer. Since, on account of the shortness of the path, the integral of every finite magnitude vanishes, we obtain the conditions—

$$(8_f)\begin{cases} Y_2 - Y_1 = \dfrac{d\phi_{1,2}}{dy}, \\ X_2 - X_1 = \dfrac{d\phi_{1,2}}{dx}; \end{cases} \qquad (8_g)\begin{cases} M_2 - M_1 = 0, \\ N_2 - N_1 = 0. \end{cases}$$

Applying these to the third of the equations $(7_a)$ and $(7_b)$, we obtain as the conditions for the normal-forces, that on both sides of the limiting surface, the values of the expressions

$$\mu_{13}\frac{dL}{dt}+\mu_{23}\frac{dM}{dt}+\mu_{33}\frac{dN}{dt},$$

$$\epsilon_{13}\frac{dX}{dt}+\epsilon_{23}\frac{dY}{dt}+\epsilon_{33}\frac{dZ}{dt}+4\pi\{\lambda_{31}(X-X') + \lambda_{32}(Y-Y')+\lambda_{33}(Z-Z')\}$$

must be equal. If the bodies on both sides of the limiting surface are homogeneous, then the presence of the electromotive forces has no effect upon the conditions by which the forces existing on the two sides are connected.

Our limiting conditions are nothing else than the general equations ($7_a$) and ($7_b$), transformed to suit special circumstances. We may, therefore, imagine every statement and every operation relating to these general equations within a definite region to be at once extended to the limits of heterogeneous bodies within the region; provided always that this procedure does not land us in mathematical impossibilities, and therefore that our statements and operations, either directly or after suitable transformation, do not cease to be finite and definite. We shall often avail ourselves of the convenience which arises from this. And if, in general, we dispense with proving that all the expressions which arise are finite and definite, it must not be supposed that this is because we regard such proof as superfluous, but only because the proof has long since been furnished, or can be supplied according to known examples, in all the cases which have to be considered.

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Each one of the previous sections means an increase in the number of facts embraced by the theory. The following sections, on the other hand, deal only with names and notations. As their introduction does not increase the number of facts embraced, they are merely accessory to the theory; their value consists partly in making possible a more concise mode of expression, and partly also in simply bringing our theory into its proper relation towards the older views as to electrical theory.

### 9. *Electric and Magnetic Polarisation*

So far as our equations relate to isotropic media, each separate one gives the value which a single one of the physical magnitudes under consideration will have at the next moment, expressed as a definite function of the conditions existing at the present moment. This form of the equations is very perfect from a mathematical point of view, for it enables us to ascertain from the outset that the equations determine without ambiguity the course of any process whatever that may be arbitrarily excited. It also appears very perfect from a much more philosophic standpoint; for it enables us to recognise on the left-hand side of the equation the future state—the consequence—while, at the same time, on the right-hand side of the equation, it exhibits the present state—the cause thereof. But those of our equations which relate to eolotropic bodies have not this perfect form; for, on the left-hand side, they do not contain the variations of a single physical magnitude, but functions of such variations. Since these functions are linear, the equations can certainly be thrown into the desired form by solving them for the separate variations. Another means to the same end—which, at the same time, simplifies the equations—is by introducing the magnitudes which we call polarisations. We put

$$(9_c)\ \begin{cases} \mathfrak{L} = \mu_{11}L + \mu_{12}M + \mu_{13}N, \\ \mathfrak{M} = \mu_{12}L + \mu_{22}M + \mu_{23}N, \\ \mathfrak{N} = \mu_{13}L + \mu_{23}M + \mu_{33}N; \end{cases} \qquad (9_d)\ \begin{cases} \mathfrak{X} = \epsilon_{11}X + \epsilon_{12}Y + \epsilon_{13}Z, \\ \mathfrak{Y} = \epsilon_{12}X + \epsilon_{22}Y + \epsilon_{23}Z, \\ \mathfrak{Z} = \epsilon_{13}X + \epsilon_{23}Y + \epsilon_{33}Z; \end{cases}$$

and call the resultant of $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ the magnetic, and the resultant of $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ the electric polarisation. For isotropic bodies the polarisations and the forces have the same direction, and the ratio of the former to the latter is the specific inductive capacity or the magnetic permeability. In the case of the ether polarisations and forces coincide. If we introduce the polarisations on the left-hand side of our equations, then each equation gives us the variation of a single polarisation-component as the result of the forces instantaneously present. Since the forces are linear functions of the polarisation, there is no difficulty in introducing the polarisations on the right-hand side of the equations as well. We should thus have replaced the particular directed magnitude—the force—which we first used to determine the electromagnetic state, by another directed magnitude—the polarisation—which would serve our purpose as well, but not any better. The introduction of the polarisations and the forces side by side considerably simplifies the equations; and this is our first indication that, in order to represent completely the conditions in ponderable bodies, it is necessary to specify at least two directed magnitudes for the electrical condition and two for the magnetic condition.

In order to simplify our equations further, let us put

$$(9_e)\begin{cases} u = \lambda_{11}(X - X') + \lambda_{12}(Y - Y') + \lambda_{13}(Z - Z'), \\ v = \lambda_{21}(X - X') + \lambda_{22}(Y - Y') + \lambda_{23}(Z - Z'), \\ w = \lambda_{31}(X - X') + \lambda_{32}(Y - Y') + \lambda_{33}(Z - Z'). \end{cases}$$

For reasons which will appear in the next section, we call $u$, $v$, $w$ the components (measured electrostatically) of the electric current-density.

Our most general equations now take the form

$$(9_a)\begin{cases} A\frac{d\mathfrak{L}}{dt} = \frac{dZ}{dy} - \frac{dY}{dz}, \\ A\frac{d\mathfrak{M}}{dt} = \frac{dX}{dz} - \frac{dZ}{dx}, \\ A\frac{d\mathfrak{N}}{dt} = \frac{dY}{dx} - \frac{dX}{dy}; \end{cases} \qquad (9_b)\begin{cases} A\frac{d\mathfrak{X}}{dt} = \frac{dM}{dz} - \frac{dN}{dy} - 4\pi A u, \\ A\frac{d\mathfrak{Y}}{dt} = \frac{dN}{dx} - \frac{dL}{dz} - 4\pi A v, \\ A\frac{d\mathfrak{Z}}{dt} = \frac{dL}{dy} - \frac{dM}{dx} - 4\pi A w, \end{cases}$$

and, on introducing the polarisations, the electromagnetic energy per unit volume of any body whatever takes the form

$$\frac{1}{8\pi}(\mathfrak{X}X + \mathfrak{Y}Y + \mathfrak{Z}Z) + \frac{1}{8\pi}(\mathfrak{L}L + \mathfrak{M}M + \mathfrak{N}N).$$

In these expressions there no longer appear any quantities which refer to any particular body. The statement that these equations must be satisfied at all points of infinite space, embraces all problems of electromagnetism; and the infinite multiplicity of these problems only arises through the fact that the constants $\epsilon$, $\mu$, $\lambda$, $X'$, $Y'$, $Z'$ of the linear relations $(9_c)$, $(9_d)$, $(9_e)$ may be functions of the space in a multiplicity of ways, varying partly continuously, and partly discontinuously, from point to point.

### 10. *Electricity and Magnetism*

Let there be a system of ponderable bodies in which electromagnetic processes take place, and which are separated by empty space from other systems. Let us differentiate the three equations ($9_b$) with respect to $x$, $y$, $z$ respectively, and add; we thus obtain for all points of the system the equation

$$\frac{d}{dt}\left(\frac{d\mathfrak{X}}{dx}+\frac{d\mathfrak{Y}}{dy}+\frac{d\mathfrak{Z}}{dz}\right) = -4\pi\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right).$$

Let us multiply this equation by the volume-element $d\tau$, and integrate over the volume up to any surface, completely enclosing the ponderable system. Let $d\omega$ be the element of this surface, and let the normal to $d\omega$ make with the axes the angles $n,x$, $n,y$, $n,z$. Since $u$, $v$, $w$ are zero at the surface, we get

$$\frac{d}{dt}\int\left(\frac{d\mathfrak{X}}{dx}+\frac{d\mathfrak{Y}}{dy}+\frac{d\mathfrak{Z}}{dz}\right)d\tau = \frac{d}{dt}\int(\mathfrak{X}\cos n,x+\mathfrak{Y}\cos n,y+\mathfrak{Z}\cos n,z)d\omega$$
$$= -4\pi\int\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)d\tau$$
$$= -4\pi\int(u\cos n,x+v\cos n,y+w\cos n,z)d\omega = 0.$$

Hence, if $e$ denotes a quantity which is independent of time—

$$(10_a)\left\{\begin{aligned} &\int\left(\frac{d\mathfrak{X}}{dx}+\frac{d\mathfrak{Y}}{dy}+\frac{d\mathfrak{Z}}{dz}\right)d\tau \\ &=\int(\mathfrak{X}\cos n,x+\mathfrak{Y}\cos n,y+\mathfrak{Z}\cos n,z)d\omega = 4\pi e.\end{aligned}\right.$$

The quantity $e$ is obviously a function of the electrical state of the system—a function of such a kind that it cannot be increased or diminished by any internal or external processes of a purely electromagnetic nature. This indestructibility of the quantity $e$—which also holds good for other than purely electromagnetic processes, so long as these are restricted to the interior of the system—has prompted the idea that $e$ represents the amount of some substance contained in the system. In accordance with this idea we call $e$ the amount of electricity contained in the ponderable system. But we must allow $e$ to be positive or negative, whereas the amount of a substance is necessarily positive. For this reason the hypothesis has been supplemented by assuming the existence of two electricities of opposite properties, and making $e$ mean the difference between the two; or else a way out of the difficulty has been sought in assuming that $e$ represents only the deviation of the amount of electricity actually contained from the normal amount. But if $e$ represents the quantity of a substance in one of these forms or some other form, then each volume-element $d\tau$ must furnish its definite share towards the total amount of $e$. Only hypothetically can we distribute the volume-integral, which supplies $e$, among the separate volume-elements. A possible distribution—the first which suggests itself for the moment—is that which assigns to the volume-element $d\tau$ the quantity of electricity—

$$\frac{1}{4\pi}\left(\frac{d\mathfrak{X}}{dx}+\frac{d\mathfrak{Y}}{dy}+\frac{d\mathfrak{Z}}{dz}\right)d\tau.$$

We shall call the quantity of electricity thus determined the true electricity of the volume-element; in the interior of a body, in accordance with this, we shall call the expression

$$\frac{1}{4\pi}\left(\frac{d\mathfrak{X}}{dx}+\frac{d\mathfrak{Y}}{dy}+\frac{d\mathfrak{Z}}{dz}\right)$$

the true volume-density, and at the surface of separation of dissimilar bodies the expression

$$\frac{1}{4\pi}\{(\mathfrak{X}_2-\mathfrak{X}_1)\cos n,x+(\mathfrak{Y}_2-\mathfrak{Y}_1)\cos n,y+(\mathfrak{Z}_2-\mathfrak{Z}_1)\cos n,z\}$$

the true surface-density of the electricity.

Another possible distribution of $e$ among the volume-elements which suggests itself is that which we get through observing that in empty space polarisations and forces are identical, and that we can therefore write instead of $(10_a)$—

$$(10_b)\begin{cases}4\pi e=\int(\mathrm{X}\cos n,x+\mathrm{Y}\cos n,y+\mathrm{Z}\cos n,z)d\omega\\ \quad\;=\int\left(\frac{d\mathrm{X}}{dx}+\frac{d\mathrm{Y}}{dy}+\frac{d\mathrm{Z}}{dz}\right)d\tau,\end{cases}$$

and furthermore, that we can regard the expression

$$\frac{1}{4\pi}\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right)d\tau$$

as representing the share which the volume-element $d\tau$ contributes to $e$. Accordingly, we call the quantity of electricity so determined the free electricity of the volume-element, and

$$\frac{1}{4\pi}\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right)$$

the free volume-density, and at surfaces of separation—

$$\frac{1}{4\pi}\{(X_2-X_1)\cos n,x+(Y_2-Y_1)\cos n,y+(Z_2-Z_1)\cos n,z\}$$

the free surface-density of the electricity. The difference between the true and the free electricity we call the bound electricity. Our nomenclature follows the familiar nomenclature which takes its origin from the view hitherto held as to the existence of electrical action-at-a-distance.[1] According to this view, a part of the extraneous or "true" amount of electricity introduced into a non-conductor remains "bound" by electrical displacement[2] in the molecules of the surrounding medium; whereas the rest remains "free" to exert its distance-action outwards. And yet in many respects our nomenclature differs from the usual one. But since the latter is sometimes ambiguous and not always consistent, it was not possible for me to find a system of notation which would in all cases harmonise with the common use of terms. The common phraseology is also ambiguous in that it uses the word electricity without further discrimination to denote sometimes the true, sometimes the free electricity; and this even when important statements are being made.

In accordance with what has been stated above, we denote the integral

$$\int(\mathfrak{X}\cos n,x+\mathfrak{Y}\cos n,y+\mathfrak{Z}\cos n,z)d\omega,$$

extended over any closed surface and divided by $4\pi$, as the

[1] [See Note 30 at end of book.]

[2] This is not identical with our polarisations. [See the theoretical part of the Introduction.]

true electricity contained within this surface. The same integral extended over an unclosed surface we shall call the number of electric lines of force traversing this surface in the direction of the positive normal. By this notation we follow Faraday's conception, according to which the lines of force are lines which in isotropic homogeneous bodies run everywhere in the direction of the prevailing force, and the number of which is proportional to the magnitude of the force. It is true that by our notation we have rendered this conception more complete or precise in this respect,—that in all bodies we make the lines of force run everywhere in the direction of the polarisation, not of the force, and that their density is in all cases proportional to the magnitude of the polarisation, not of the force. It follows from our definitions that the quantity of true electricity contained in any space, multiplied by $4\pi$, is equal to the excess of the number of lines of force which enter the surface over the number which leave it. Every line of force which has an end must accordingly end in true electricity; and we may define the true electricity as the free ends of the lines of force. If a given space in the neighbourhood of the surface over which our integral extends is free from true electricity, then the value of the integral is independent of the particular position of the surface within this space; it only depends upon the position of the boundary of the surface. In this case, then, we denote the value of the integral as being the number of lines of force crossing the boundary line—any ambiguity remaining in this expression being supposed removed by special restrictions.

We shall next calculate the variation of the true electricity $e_w$ in a part of our system bounded in any way. Let $d\omega$ again be an element of the bounding surface of this part. We get

$$(10_c)\quad \frac{de_w}{dt} = -\int\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)d\tau = -\int(u\cos n,x + v\cos n,y + w\cos n,z)d\omega.$$

Now if our bounding surface runs entirely in bodies for which $\lambda$ is equal to zero, then $u$, $v$, $w$ still vanish at the surface, and hence the amount of true electricity contained in the space bounded by it remains constant. Accordingly, true electricity cannot by any purely electromagnetic process escape from a space which is wholly bounded by bodies for which $\lambda$ is equal to zero. For this reason we call, and have called, such bodies non-conductors. But if the bounding surface passes wholly or partially through bodies for which $\lambda$ is not zero, it becomes possible for the amount of electricity within the space so bounded to vary through purely electrical disturbances; for this reason we call bodies of this latter kind conductors. This division of bodies into conductors and non-conductors has reference therefore to the true electricity; with reference to the free electricity all bodies may be regarded as conductors (cf. displacement-currents). The amount of a substance within a given space can only vary by its passing inwards or outwards through the surface; and it is clear that a definite amount of the substance must pass through each element of the surface. Consistently with the fact that the amount of electricity given by our integral passes per unit time through every closed surface, we may assume that the amount

$$u \cos n,x + v \cos n,y + w \cos n,z$$

passes through unit surface of every surface-element. In accordance with this assumption we call, and have called, $u$, $v$, $w$ the components of the electric current-density, and the integral

$$\int(u \cos n,x + v \cos n,y + w \cos n,z)d\omega,$$

taken over an unclosed surface, the electric current flowing through this surface. We must, however, lay stress upon this—that even if we admit the materiality of electricity, the above special determination of its flow in conductors embraces a further hypothesis. Upon the system of disturbance found there can be superposed an arbitrary current-system, closed at every moment, without thereby altering the increase or decrease of electricity at any point.

If a portion of our system has attained its present condition, starting from the unelectrified condition, by purely electromagnetic processes, or if by purely electromagnetic changes it can return to the unelectrified state, then in all non-conductors of this portion the true electricity is equal to zero. For such portions we have, then, in addition to the general equations, the following as limitations of the permissible initial conditions which are not inconsistent with the general equations, viz. :—

$$\frac{d\mathfrak{X}}{dx}+\frac{d\mathfrak{Y}}{dy}+\frac{d\mathfrak{Z}}{dz}=0$$

for the interior of non-conductors; and

$$(\mathfrak{X}_2-\mathfrak{X}_1)\cos n,x+(\mathfrak{Y}_2-\mathfrak{Y}_1)\cos n,y+(\mathfrak{Z}_2-\mathfrak{Z}_1)\cos n,z=0$$

for the boundary between two heterogeneous non-conductors.

The magnetic phenomena can be considered in a manner exactly analogous to the electric phenomena. Let us proceed to examine these, with the assistance of the equations ($9_a$). We shall call

$$\frac{1}{4\pi}\left(\frac{d\mathfrak{L}}{dx}+\frac{d\mathfrak{M}}{dy}+\frac{d\mathfrak{N}}{dz}\right)$$

the true volume-density for the interior of a body; the expression—

$$\frac{1}{4\pi}\{(\mathfrak{L}_2-\mathfrak{L}_1)\cos n,x+(\mathfrak{M}_2-\mathfrak{M}_1)\cos n,y+(\mathfrak{N}_2-\mathfrak{N}_1)\cos n,z\}$$

the true surface-density of magnetism at the surface of separation of two bodies; and the integral of these magnitudes extended over a definite portion of space, the true magnetism contained in this portion. The integral

$$\int(\mathfrak{L}\cos n,x+\mathfrak{M}\cos n,y+\mathfrak{N}\cos n,z)d\omega,$$

taken over an unclosed surface, we shall call the number of magnetic lines of force penetrating this surface, or the boundary of this surface. Further, we shall call

$$\frac{1}{4\pi}\left(\frac{dL}{dx}+\frac{dM}{dy}+\frac{dN}{dz}\right)$$

the free volume-density for the interior of a body; and

$$\frac{1}{4\pi}\{(L_2-L_1)\cos n,x+(M_2-M_1)\cos n,y+(N_2-N_1)\cos n,z\}$$

the free surface-density of the magnetism at the surface of separation of two bodies. The distinction between conductors and non-conductors here disappears; for the equations ($9_a$) contain no terms corresponding to the $u$, $v$, $w$ of equations ($9_b$). With respect to true magnetism all bodies are non-conductors;

with respect to free magnetism all bodies may be regarded as conductors.

Let us suppose that a system or a portion of it emerges from the non-magnetic state through purely electromagnetic processes, or can by such processes return to this state. For such a system, or that portion of it, the following equations obtain, viz.—

$$\frac{d\mathfrak{L}}{dx}+\frac{d\mathfrak{M}}{dy}+\frac{d\mathfrak{N}}{dz}=0$$

for the interior of the bodies; and

$$(\mathfrak{L}_2-\mathfrak{L}_1)\cos n,x+(\mathfrak{M}_2-\mathfrak{M}_1)\cos n,y+(\mathfrak{N}_2-\mathfrak{N}_1)\cos n,z=0$$

for the surface of separation of heterogeneous bodies. These are supplementary to the general equations as consistent stipulations respecting the possible initial conditions.

## 11. *Conservation of Energy*

Let S denote the electromagnetic energy of a volume $\tau$, which is bounded by the surface $\omega$. We can calculate the variation of S by multiplying all the equations ($9_a$) and ($9_b$) by $(1/4\pi A)d\tau$, then multiplying them separately in order by L, M, N, X, Y, Z, adding all together, and integrating over the volume $\tau$. We obtain

$$(11_a)\begin{cases}\dfrac{dS}{dt}=\dfrac{1}{4\pi A}\displaystyle\int\{(NY-MZ)\cos n,x+(LZ-NX)\cos n,y\\ \quad+(MX-LY)\cos n,z\}d\omega-\int(uX+vY+wZ)d\tau.\end{cases}$$

If we extend the space $\tau$ over a complete electromagnetic system, *i.e.* as far as a surface at which the forces vanish, then our equation becomes

$$\frac{dS}{dt}=-\int(uX+vY+wZ)d\tau.$$

The conservation of energy accordingly requires that in every system which is not subjected to external actions, an amount of energy corresponding to the integral on the right-hand side should make its appearance per unit time in other than electromagnetic form. This requirement is satisfied by experience, which further teaches us that each separate volume-element $d\tau$ furnishes towards the total amount of the transformed energy the quantity

$$(uX + vY + wZ)d\tau,$$

and shows us in what form this energy makes its appearance. Or rather, to speak accurately, experience does not show that this is true in all cases, but provisionally in the following special cases only. According to both theory and experience, the amount of energy which appears per unit time and per unit volume in the interior of a homogeneous isotropic conductor takes the form

$$\lambda(X^2 + Y^2 + Z^2) = \frac{1}{\lambda}(u^2 + v^2 + w^2).$$

It is always positive and represents a development of heat—the Joule effect. At the boundary between two homogeneous isotropic bodies, the amount of energy per unit volume that appears in the transition-layer takes the form

$$uX' + vY' + wZ';$$

hence, by integration over the whole thickness of the transition-layer, it follows that the quantity of energy which appears per unit of surface at the boundary amounts to

$$(u \cos n,x + v \cos n,y + w \cos n,z) \,.\, \phi_{1,2},$$

which expression is also confirmed by experience. This expression may be either positive or negative; it may correspond either to an appearance or a disappearance of foreign forms of energy. Either the transformed foreign energy is heat in this case as well—the Peltier effect; in which case we denote the effective electromotive forces as thermoelectric. Or else chemical energy as well as heat is transformed; in which case we denote the forces as electrochemical. Let us now consider any limited portion of our system and calculate for it the increase of its total energy, *i.e.* of the quantity

$$\frac{dS}{dt} + \int (uX + vY + wZ)d\tau.$$

In accordance with what has been stated, we find that this increase is equal to an integral taken over the surface of the space. The variation of the stock of energy in this (and therefore in any) space will therefore be correctly calculated if we assume that the energy enters after the manner of a substance through the surface, and in such quantity that through every such surface the amount

$$\frac{1}{4\pi A}\{(NY-MZ)\cos n,x+(LZ-NX)\cos n,y + (MX-LY)\cos n,z\}$$

enters per unit of surface. A geometrical discussion of this expression shows that our assumption is identical with the statement that the energy moves everywhere in a direction perpendicular to the directions of the magnetic and electric forces, and in such amount that in this direction a quantity equal to the product of the two forces, the sine of the enclosed angle, and the factor $1/4\pi A$, passes through unit surface per unit time. This is Dr. Poynting's highly remarkable theory on the transfer of energy in the electromagnetic field.[1] In examining its physical meaning we must not forget that our analysis of the surface-integral into its elements was hypothetical, and that the result thereof is not always probable. If a magnet remains permanently at rest in presence of an electrified body, then in accordance with this result the energy of the neighbourhood must find itself in a state of continuous motion, going on, of course, in closed paths. In the present state of our knowledge respecting energy there appears to me to be much doubt as to what significance can be attached to its localisation and the following of it from point to point. Considerations of this kind have not yet been successfully applied to the simplest cases of transference of energy in ordinary mechanics; and hence it is still an open question whether, and to what extent, the conception of energy admits of being treated in this manner.[2]

## 12. *Ponderomotive Forces*

The mechanical forces, which we perceive between ponderable bodies in the electromagnetically stressed field, we regard

[1] J. H. Poynting, *Phil. Trans.* **2**, p. 343, 1884.
[2] [See Note 31 at end of book.]

as the resultants of mechanical pressures which are excited by the existence of electromagnetic stresses in the ether and in other bodies. According to this view the mechanical forces which act upon a ponderable body are completely determined by the electromagnetic state of its immediate neighbourhood; and it need not be considered what causes at a distance may have led up to this state. We further assume that the presupposed pressures are of such a kind that they cannot give rise to any resultants which would tend to set the interior of the ether itself in motion. Without this hypothesis our system would necessarily be incorrect, or at least incomplete; for without it we could not in general speak at all of electromagnetic forces in the ether at rest. It necessarily follows from this hypothesis that the forces under observation, acting upon ponderable bodies, must satisfy the principle of the equality of action and reaction.

The question now is—Whether pressures can be specified answering these requirements, and capable of producing the resultants which are actually observed? Maxwell, and, in a more general form, von Helmholtz have described forms of pressures which satisfy all the requirements of statical and stationary states. But these pressures, if assumed to obtain for the general variable state, would set the ether itself in motion. We therefore assume that the complete forms have not been discovered, and, avoiding any definite statements as to the magnitude of the pressures, we shall rather deduce the ponderomotive forces with the aid of the hypotheses already stated, of the principle of the conservation of energy, and of the following fact derived from experience:—If the ponderable bodies of an electrically or magnetically excited system, which always remains indefinitely near to the statical condition, are displaced with reference to one another, and if at the same time the amount of true electricity and of true magnetism in each element of the bodies remains invariable and behaves as if attached to the element, then the mechanical work consumed in the displacement of the bodies finds its only compensation in the increase of the electromagnetic energy of the system, and is therefore equal to this latter.[1]

It still remains an open question whether forms of pressure can be specified which satisfy, generally and precisely, the

[[1] See Note 32 at end of book.]

requirements which we have laid down. If this is not the case, our body of hypotheses contains an intrinsic contradiction which must be removed by correcting one or more of these hypotheses. But at all events the necessary amendments are of such a kind that their effect would not make itself felt in any of the phenomena hitherto observed. And it must be pointed out that if there is here something lacking in our theory, it is not a defect in the foundations of the theory, but in parts of the superstructure. For, from our point of view, the mechanical forces excited are secondary consequences of the electromagnetic forces. We could discuss the theory of the latter without even mentioning the former; as indeed we have excluded from the discussion all other phenomena of minor importance which result from the electromagnetic state.

## B. Deduction of the Phenomena from the Fundamental Equations

We divide the phenomena represented by our equations into statical, stationary, and dynamical. In order that a phenomenon may rank as statical or stationary, it is necessary that it should not determine any variations of the electric and magnetic forces with time, *i.e.* that the left-hand sides of the equations ($9_a$) and ($9_b$) should vanish. Furthermore, in order that a phenomenon may rank as statical, it is also necessary that it should not be accompanied by changes in time at all, and hence, more especially, that it should not determine any permanent change of energy into other forms. The sufficient and necessary condition for this is that the quantities $u, v, w$ in equations ($9_a$) and ($9_b$) should also vanish.

### Statical Phenomena

If in the equations ($9_a$) and ($9_b$) the left-hand sides and also the quantities $u, v, w$ vanish, the system splits up into two mutually independent systems, of which one contains only the electric forces and the other only the magnetic forces. We thus get two groups of problems, of which one is called electrostatics, and the other might be called magnetostatics.

## 13. *Electrostatics*

In this section we shall disregard the occurrence of electromotive forces; for if these admit of the existence of the statical state at all, their action is too weak to come into consideration in the problems which are of interest. In conductors, accordingly, in which the quantities $\lambda$ do not vanish, the forces X, Y, Z must vanish. In non-conductors the equations ($9_a$) take the form

$$(13_a) \quad \frac{dZ}{dy} - \frac{dY}{dz} = \frac{dX}{dz} - \frac{dZ}{dx} = \frac{dY}{dx} - \frac{dX}{dy} = 0.$$

Hence the forces possess a potential $\phi$, and can be put equal to the negative differential coefficient of this potential. Since the forces are everywhere finite, $\phi$ is everywhere continuous; it can therefore continue right through the conductors, and is then to be regarded as constant within these. At a surface of separation the differential coefficients of $\phi$ tangential to the separating surface continue through it without discontinuity. Again, if $e_f$ denote the volume-density of the free electricity, according to section (10) the potential $\phi$ satisfies everywhere in space the equation $\Delta\phi = -4\pi e_f$. In free ether this assumes the form $\Delta\phi = 0$; and after suitable transformation for the surface of separation between heterogeneous bodies it gives the condition

$$\left(\frac{d\phi}{dn}\right)_2 - \left(\frac{d\phi}{dn}\right)_1 = -4\pi e_f,$$

where $e'_f$ denotes the surface-density of the free electricity. From all these conditions it follows that the value of $\phi$, within an arbitrary constant, is definite and equal to $\int(e_f/r)d\tau$, the integral being extended over the whole space with due regard to the surfaces of separation. Thus when the potential and the forces are distributed in the same way in different non-conductors, the free electricities are the same. But the corresponding quantities of true electricity are different, and for the interior of two homogeneous non-conductors they are in the ratio of the specific inductive capacities. Restricting ourselves for the moment to isotropic bodies, the condition that the density of the true electricity in the interior of the non-conductors should have given values $e_w$, is expressed by the equation

$$\frac{d}{dx}\left(\epsilon\frac{d\phi}{dx}\right)+\frac{d}{dy}\left(\epsilon\frac{d\phi}{dy}\right)+\frac{d}{dz}\left(\epsilon\frac{d\phi}{dz}\right)=-4\pi e_w,$$

which at the boundary of two isotropic bodies assumes the form

$$\epsilon_2\left(\frac{d\phi}{dn}\right)_2-\epsilon_1\left(\frac{d\phi}{dn}\right)_1=-4\pi e'_w,$$

where $e'_w$ denotes the surface-density of the true electricity.

Let us now direct our attention to the stock of energy in an electrostatic system. We obtain this successively in the forms

$$\frac{1}{8\pi}\int(\mathfrak{X}X+\mathfrak{Y}Y+\mathfrak{Z}Z)d\tau=-\frac{1}{8\pi}\int\left(\mathfrak{X}\frac{d\phi}{dx}+\mathfrak{Y}\frac{d\phi}{dy}+\mathfrak{Z}\frac{d\phi}{dz}\right)d\tau$$

$$=\frac{1}{8\pi}\int\phi\left(\frac{d\mathfrak{X}}{dx}+\frac{d\mathfrak{Y}}{dy}+\frac{d\mathfrak{Z}}{dz}\right)d\tau=\tfrac{1}{2}\int\phi e_w d\tau=\tfrac{1}{2}\int\int\frac{e_w e_f}{r}d\tau d\tau.$$

The integrations are here supposed to extend over all space in which electrical stresses exist, and therefore up to the boundaries where the stresses vanish, and the suitable transformation of the integrals at the bounding surfaces is implicitly assumed. When any motion of the ponderable bodies takes place, and the amounts of true electricity attached to the elements of these bodies remain constant, then, according to section (12), the increase in the value of any one of these expressions is equal to the work done by the mechanical forces in this motion. Hence, if our system consists of two quantities of electricity $E_1$ and $E_2$, separated by the ether and at a distance R apart which is very great compared with their own dimensions, and if their distance apart increases by an amount $dR$, the electric energy of the space decreases by an amount

$$\tfrac{1}{2}(E_1E_2+E_2E_1)\frac{dR}{R^2}.$$

Thus the expression $E_1E_2/R^2$ represents the mechanical force with which the two electricities tend to move apart.

Coulomb's law, which, in the older theories forms the starting-point of every discussion, here makes its appearance as a remote final result.

With regard to the general determination of the ponderomotive forces, we must here content ourselves with the following remarks:—The last two expressions obtained for the energy are just those whose variations represent the work done by the motion of bodies in ordinary electrostatics. Hence it follows that from the variations of these expressions we can calculate the values of those same forces which are the starting-point of ordinary electrostatics and are tested by experiment. In particular, it may be shown that an element of a body which contains a quantity $e$ of true electricity is acted upon by the mechanical force-components $e$X, $e$Y, $e$Z. We thus return to the same statements by means of which we first introduced the electric forces.

## 14. *Magnetostatics*

The equations which connect the components of statical magnetic forces are the same as those which obtain between the components of statical electric forces. Hence all the statements in the preceding section may, with the necessary changes of notation, be repeated here. And if, nevertheless, the magnetic problems of interest are still distinct mathematically from the electrostatic problems, this arises from the following causes:—

(1) The class of bodies known as conductors is here wanting.

(2) In no bodies, excepting those which exhibit permanent or remanent magnetism, does true magnetism appear. Hence in the interior of such bodies, provided they are isotropic, the magnetic potential $\psi$ must necessarily and always satisfy the equation

$$\frac{d}{dx}\left(\mu\frac{d\psi}{dx}\right)+\frac{d}{dy}\left(\mu\frac{d\psi}{dy}\right)+\frac{d}{dz}\left(\mu\frac{d\psi}{dz}\right)=0,$$

which at the boundary between two such bodies becomes

$$\mu_2\left(\frac{d\psi}{dn}\right)_2-\mu_1\left(\frac{d\psi}{dn}\right)_1=0.$$

The equations which apply to the interior and the boundaries of crystalline bodies are somewhat more complicated, but can easily be given; these equations have to be considered in discussing the phenomena of the so-called magne-crystallic force.

(3) The specific inductive capacity of all known bodies is greater than unity; on the other hand, the magnetic permeability of many bodies is less than unity. We call such bodies diamagnetic, and all others paramagnetic. The free magnetic density at the surface of an isotropic body bounded by empty space is equal to $(1-\mu)$ times the force in the interior of the body normal to the surface. The sign of the surface-magnetism (*Belegung*) of a diamagnetic body is therefore opposite to that of a paramagnetic body when the sense of the force is the same.

The study of statical magnetism further acquires a peculiar aspect, owing to the fact that iron and steel, which are the most important substances in connection with magnetic phenomena, do not fit in at all well with the theoretical treatment. These substances exhibit permanent and remanent magnetism; hence the polarisation of the ponderable material is here partly independent of the prevailing force, and therefore the magnetic state cannot be completely defined by a single directed magnitude. Again, the relations between the force and the disturbances produced by it are not linear; so that, for a double reason, our theory does not include these bodies entirely within its scope. In order to avoid excluding them entirely from consideration, we replace them by the two ideal substances which approximate most nearly to them—perfectly soft iron and perfectly hard steel. We define the first as a substance which obeys our equations, and for which the value of $\mu$ is very large. We attain a nearer approximation by giving $\mu$ different values according to the problem under consideration. We define perfectly hard steel as a substance which obeys our equations, whose magnetic permeability is unity, in whose interior true magnetism can exist distributed in any way, provided always that the total quantity of true magnetism existing in any such piece of steel does not differ from zero.

## Stationary States

The same conditions hold good for the state of stationary disturbances in non-conductors as for the statical condition; in conductors, which for the sake of simplicity we shall assume in this section to be isotropic, the equations ($9_a$), ($9_b$), ($9_c$), which here come under consideration, take the form

$$(15_a)\left\{\begin{aligned}&\frac{dZ}{dy}-\frac{dY}{dz}=0,\\&\frac{dX}{dz}-\frac{dZ}{dx}=0,\\&\frac{dY}{dx}-\frac{dX}{dy}=0;\end{aligned}\right.\qquad(15_b)\left\{\begin{aligned}&\frac{dM}{dz}-\frac{dN}{dy}=4\pi Au,\\&\frac{dN}{dx}-\frac{dL}{dz}=4\pi Av,\\&\frac{dL}{dy}-\frac{dM}{dx}=4\pi Aw.\end{aligned}\right.$$

$$(15_c)\quad u=\lambda(X-X'),\quad v=\lambda(Y-Y'),\quad w=\lambda(Z-Z').$$

Differentiating equations ($15_b$) with respect to $x$, $y$, $z$ respectively, and adding, we get

$$(15_d)\quad \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0,$$

which equation, at surfaces where the currents vary abruptly, takes the form

$$(15_e)\quad (u_2-u_1)\cos n,x+(v_2-v_1)\cos n,y+(w_2-w_1)\cos n,z=0.$$

Combining equations ($15_d$) and ($15_e$) with equations ($15_a$) and ($15_c$), we obtain a system which contains only the electric forces. This can be treated without regard to the magnetic forces, and gives us the theory of current-distribution. If the components $u$, $v$, $w$ of the current are found, the treatment of the equations ($15_b$) further gives us the magnetic forces exerted by these currents.

### 15. *Distribution of Steady Currents*

It appears from equations ($15_a$) that, in the interior of the conductor through which a current is flowing, the forces can also be represented as the negative differential coefficients of a function $\phi$, the potential, which is determined by the following condition, which must obtain everywhere :—

$$(15_f)\left\{\begin{aligned}\frac{d}{dx}\Big(\lambda\frac{d\phi}{dx}\Big)+\frac{d}{dy}\Big(\lambda\frac{d\phi}{dy}\Big)+\frac{d}{dz}\Big(\lambda\frac{d\phi}{dz}\Big)=&-\frac{d}{dx}(\lambda X')\\&-\frac{d}{dy}(\lambda Y')-\frac{d}{dz}(\lambda Z').\end{aligned}\right.$$

At the surface separating two heterogeneous conductors this equation takes the form

$$(15_g)\quad \left\{ \begin{array}{l} \lambda_2\left(\frac{d\phi}{dn}\right)_2 - \lambda_1\left(\frac{d\phi}{dn}\right)_1 = -(\lambda_2 X'_2 - \lambda_1 X'_1)\cos n,x \\ -(\lambda_2 Y'_2 - \lambda_1 Y'_1)\cos n,y - (\lambda_2 Z'_2 - \lambda_1 Z'_1)\cos n,z, \end{array} \right.$$

and hence at the boundary between a conductor and a non-conductor the form

$$(15_h)\quad \frac{d\phi}{dn} = -X'\cos n,x - Y'\cos n,y - Z'\cos n,z.$$

In addition to these limiting conditions we have, according to section (8), at limiting surfaces where the electromotive forces become infinite, the further condition

$$(15_i)\quad \left\{ \begin{array}{l} \phi_1 - \phi_2 = \int(X\cos n,x + Y\cos n,y + Z\cos n,z)dn, \\ \qquad\quad = \int(X'\cos n,x + Y'\cos n,y + Z'\cos n,z)dn, \\ \qquad\quad = \phi_{1,2}\,. \end{array} \right.$$

These conditions together determine $\phi$ definitely within a constant which remains dependent upon the conditions outside the conductor. For homogeneous conductors the equations $(15_f)$ to $(15_i)$ assume the simpler forms—

$$(15_k)\quad \left\{ \begin{array}{ll} \Delta\phi = 0 & \text{for the interior of the conductor,} \\ \lambda_1\left(\frac{d\phi}{dn}\right)_1 = \lambda_2\left(\frac{d\phi}{dn}\right)_2 & \text{for the boundary between two conductors,} \\ \frac{d\phi}{dn} = 0 & \text{for the boundary adjoining a non-conductor,} \\ \phi_1 - \phi_2 = \phi_{1,2} & \text{at a bounding surface where electromotive effects occur.} \end{array} \right.$$

The equations thus obtained admit of immediate application to problems on current-distribution in bodies of three dimensions. Their application to lamellar conductors or to linear conductors is easy, and gives the definition of resistance, Ohm's law for closed circuits, Kirchhoff's laws for branched circuits, as well as the other laws relating to the distribution of steady currents.

## 16. *Magnetic Forces of Steady Currents*

In order to determine the forces L, M, N produced by the current-components $u$, $v$, $w$, which are now known, we introduce as subsidiary magnitudes the so-called components of the vector-potential, putting

$$\mathrm{U} = \int \frac{u}{r} d\tau, \qquad \mathrm{V} = \int \frac{v}{r} d\tau, \qquad \mathrm{W} = \int \frac{w}{r} d\tau.$$

The integrals are to be extended over the whole space; thus it follows from the conditions of the steady state that

$$\frac{d\mathrm{U}}{dx} + \frac{d\mathrm{V}}{dy} + \frac{d\mathrm{W}}{dz} = 0.$$

We now put

$$(16_a) \begin{cases} \mathrm{L} = \mathrm{A}\left(\dfrac{d\mathrm{V}}{dz} - \dfrac{d\mathrm{W}}{dy}\right), \qquad \mathrm{M} = \mathrm{A}\left(\dfrac{d\mathrm{W}}{dx} - \dfrac{d\mathrm{U}}{dz}\right), \\ \mathrm{N} = \mathrm{A}\left(\dfrac{d\mathrm{U}}{dy} - \dfrac{d\mathrm{V}}{dx}\right). \end{cases}$$

These quantities L, M, N are solutions of equations ($15_b$), and satisfy the equation

$$\frac{d\mathrm{L}}{dx} + \frac{d\mathrm{M}}{dy} + \frac{d\mathrm{N}}{dz} = 0.$$

If, therefore, the forces actually present differ from these, the differences between the two still satisfy the conditions for the forces of statical magnetism, and may be regarded as arising from these latter; this, however, does not exclude the supposition that the magnetism itself is due to currents. But if no statical magnetism is present at all, the formulæ above given represent completely the magnetic forces present.

If we have only to deal with linear conductors, in which the current $i$ flows, then the expressions $ud\tau$, $vd\tau$, $wd\tau$ in the quantities U, V, W are replaced by the expressions $idx$, $idy$, $idz$, where $dx$, $dy$, $dz$ are the projections of the element $ds$ of the circuit on the three axes; and the integrations must then be taken round all the circuits. Suppose we wish to regard the magnetic forces of the whole current as the sum of the actions of the separate current-elements. In order to simplify the formulæ, let us suppose the element to be at the origin and the point $x' y' z'$ to be in the $xy$-plane; then an analysis of our integrals, which, as far as its results are concerned, is admissible, gives for the action of the current-element $idx$ upon the point $x' y' z'$—

$$L = 0, \quad M = 0, \quad N = Aidx \frac{d\frac{1}{r}}{dy'} = -\frac{Aidx}{r^2} \cdot \frac{y'}{r},$$

which formulæ contain the expression of Ampère's rule and the Biot-Savart law.

Wherever $u, v, w$ vanish, *i.e.* everywhere outside the conductor in which the current flows, the values of the forces must in accordance with equations ($15_b$) possess a potential $\Psi$, to whose negative differential coefficients we can equate them. If the forces arise from only a single closed linear circuit, this potential can be expressed in the form

$$(16_b) \qquad \Psi = -Ai\int \frac{d\frac{1}{r}}{dn} d\omega + \text{const.}$$

where $d\omega$ denotes the element of any surface through the circuit, $n$ the normal to this surface, and where the integration is extended over the whole of the surface bounded by the circuit. We here regard as positive that side of the surface from which the current appears to flow in the direction in which the hands of a clock move. For the negative differential coefficients of the above expression can in all cases be brought, by known methods of transforming integrals, into the forms found for L, M, N. Except in the circuit itself these differential coefficients are therefore everywhere finite and continuous; and, even if the integral contained in $\Psi$ becomes discontinuous at the surface $\omega$, the necessary continuity can always be conferred upon $\Psi$ itself if we regard the constant contained in it as having an infinite number of values, and employ a value varying by $4\pi Ai$ whenever we pass through the surface $\omega$. The potential itself thus attains an infinite number of values, and changes in value by $4\pi Ai$ each time we return to the same point after passing round the circuit.

Various interpretations can be assigned to the integral which occurs in $\Psi$. In the first place, it can be regarded as the potential due to a magnetic shell. By following out this conception we arrive at Ampère's theory of magnetism. Again we may, with Gauss, regard the value of this integral at a given point as the spherical angle which the circuit subtends at this point. From this, by an easy transition, we arrive at the following statement:—For any given point this integral represents the number of lines of force which proceed from an unit pole placed at the point and are embraced by the circuit. We may supplement this by applying the following statement to the potential itself (including its manifoldness):—The difference between its values at two points is equal to the product of $Ai$ into the number of lines of force which cut the circuit in a definite direction when an unit pole is moved in any manner from the one point to the other.

From our standpoint the last interpretation is the most suitable; it also allows us, with the aid of sections (12) and (14), to deduce the following conclusions:—Firstly, the mechanical work which must be done in moving a magnet-pole, or a system of unchangeable magnetism, in the neighbourhood of a current whose strength is kept constant, is equal to the number of lines of force of the magnet-pole or magnetic system which cut the circuit in a definite direction, multiplied by the current and the constant A. Secondly, the mechanical work which must be done in moving a constant current in a magnetic field is equal to the number of lines of force which are cut by the circuit during the motion, multiplied by the current and the constant A. Lastly, and in particular, the mechanical work which must be done in moving a constant current 1 in the neighbourhood of a constant current 2, is equal to the number of lines of force proceeding from the circuit 2 which are cut by the circuit 1 during the motion, multiplied by the current in 1 and by the constant A. It is also equal to the number of lines of force proceeding from the circuit 1 which cut the circuit 2 during the motion, multiplied by the current in 2 and by the constant A. Both expressions lead to the same result; we can prove this by representing the product of the current in the one circuit and the number of lines of force from the other circuit which pass through it, by an expression which is symmetrical with reference to both. For let the symbols $i$, $ds$ refer to the circuit 1; and the symbols $i'$, $ds'$, U′, V′, W′, L′, M′, N′ refer to the circuit 2. Then the product of A$i$ into the number of lines of force from 2 which pass through 1 is equal to

$$\begin{aligned} &Ai\int(L'\cos n,x + M'\cos n,y + N'\cos n,z)d\omega \\ &= A^2 i\int\left\{\left(\frac{dV'}{dz}-\frac{dW'}{dy}\right)\cos n,x + \left(\frac{dW'}{dx}-\frac{dU'}{dz}\right)\cos n,y \right. \\ &\qquad \left. + \left(\frac{dU'}{dy}-\frac{dV'}{dx}\right)\cos n,z\right\} d\omega \\ &= -A^2 i\int(U'\cos s,x + V'\cos s,y + W'\cos s,z)ds \\ &= -A^2 ii'\int\int\frac{\cos s,x\cos s',x + \cos s,y\cos s',y + \cos s,z\cos s',z}{r}dsds' \\ &= -A^2 ii'\int\int\frac{\cos\epsilon}{r}dsds', \end{aligned}$$

where $\epsilon$ denotes the angle between the two current-elements. The expression obtained is symmetrical with respect to both circuits. We know that in fact the variations of this expression—Neumann's potential of the one circuit upon the other multiplied by $A^2 ii'$—gives the work required for the relative displacement of closed currents, and hence the ponderomotive forces which exist between the currents when at rest. We also know that this statement contains everything that can with certainty be asserted respecting the ponderomotive forces which arise between currents.

We shall next calculate the magnetic energy of a space in which the stationary current-components $u, v, w$ and the unchangeable magnetic densities $m$ are distributed, assuming the restriction that no magnetisable bodies are present in the space. If $\Psi$ now represents the potential of the magnetisms $m$, we obtain the energy successively in the forms—

$$(16_c)\left\{\begin{aligned} &\frac{1}{8\pi}\int(L^2+M^2+N^2)d\tau \\ &= \frac{A}{8\pi}\int\left\{L\left(\frac{dV}{dz}-\frac{dW}{dy}-\frac{1}{A}\frac{d\Psi}{dx}\right) + M\left(\frac{dW}{dx}-\frac{dU}{dz}-\frac{1}{A}\frac{d\Psi}{dy}\right)\right. \\ &\qquad \left. + N\left(\frac{dU}{dy}-\frac{dV}{dx}-\frac{1}{A}\frac{d\Psi}{dz}\right)\right\}d\tau \end{aligned}\right.$$

$$
(16_c)\left\{\begin{aligned}
&= \frac{A}{8\pi}\int\Big\{U\Big(\frac{dM}{dz}-\frac{dN}{dy}\Big)+V\Big(\frac{dN}{dx}-\frac{dL}{dz}\Big)\\
&\qquad\qquad +W\Big(\frac{dL}{dy}-\frac{dM}{dx}\Big)\Big\}d\tau\\
&\qquad\qquad +\frac{1}{8\pi}\int\Psi\Big(\frac{dL}{dx}+\frac{dM}{dy}+\frac{dN}{dz}\Big)d\tau\\
&= \tfrac{1}{2}A^2\int(Uu+Vv+Ww)d\tau+\tfrac{1}{2}\int\Psi m d\tau,\\
&\text{or, in the case of linear currents—}\\
&= \tfrac{1}{2}A^2\iint\frac{ii'\cos\epsilon}{r}\,ds ds'+\tfrac{1}{2}\int\Psi m d\tau,
\end{aligned}\right.
$$

where, in the first part of the last form, the integration is to be carried out with respect to both $ds$ and $ds'$, and is to include all currents present. It is clear from this last form that the displacement of unchangeable magnets with respect to unchangeable currents does not alter the magnetic energy of the space. Hence the mechanical work which is done in such a displacement does not find its compensation in the variation of the magnetic energy of the space, as it does in the case of the displacement of unchangeable magnets among themselves; we must account in some other way for the work which has been done. It further appears from the same formula that the relative displacement of currents which are maintained constant does determine a change in the energy of the space, which is equal to the absolute value of the work done. But when we pay due regard to the signs, we see that this change does not take place in such a sense that it can be regarded as the compensation for the lost mechanical energy, but in the opposite sense. Here again, then, we have to account for double the amount of work which the mechanical forces do in the relative displacement of the circuits. We shall return to this at the end of the following section.

## Dynamical Phenomena

From among the infinite number of possible forms of the variable state, comparatively few groups of phenomena have hitherto fallen under observation. We shall refer to these groups without attempting any exhaustive and systematic classification of the subject.

### 17. *Induction in Closed Circuits*

In accordance with equations ($9_a$) electric forces must necessarily be present in a varying magnetic field. In general these forces must be very weak, for they contain the very small factor A; on this account they can only be detected through the currents which they excite in closed circuits, or through their cumulative action in very long linear circuits which are closed to within a small fraction of their lengths. Hence the effects which can be experimentally measured invariably give us only the integral effect of the electric force in a closed circuit, *i.e.* the integral $\int(\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz)$ taken along a looped line. According to a known method of transforming integrals, which we have already used, this line-integral is equal to the surface-integral

$$\int\left\{\left(\frac{d\mathrm{Z}}{dy} - \frac{d\mathrm{Y}}{dz}\right)\cos n,x + \left(\frac{d\mathrm{X}}{dz} - \frac{d\mathrm{Z}}{dx}\right)\cos n,y + \left(\frac{d\mathrm{Y}}{dx} - \frac{d\mathrm{X}}{dy}\right)\cos n,z\right\}d\omega,$$

taken over any surface $\omega$ bounded by the line in question. Applying equations ($9_a$) this expression becomes equal to

$$\mathrm{A}\frac{d}{dt}\int(\mathfrak{L}\cos n,x + \mathfrak{M}\cos n,y + \mathfrak{N}\cos n,z)d\omega.$$

We may express this in words as follows:—The electromotive force which manifests itself in a closed circuit is equal to the variation per unit time of the number of magnetic lines of force which traverse the circuit multiplied by A. In particular, if the induction arises from a closed variable current, and if it is assumed that no magnetisable bodies are present, then according to the results of the previous section the induced electromotive force is equal to the product of the Neumann's potential of the two circuits on one another and the variation per unit time of the inducing current, multiplied by $\mathrm{A}^2$. These laws—of which the first is the more general—with their consequences embrace all the phenomena of induction which have been actually observed in the case of conductors at rest.

Induction in moving conductors lies beyond the range to which the present dissertation is restricted. But as far as linear conductors are concerned, the transition from the case of induction in conductors at rest can be made by the following statement:—Whether the magnetic field in the immediate neighbourhood of a closed circuit changes in consequence of the motion of ponderable bodies, or in consequence of purely electromagnetic changes of state, the electromotive force produced in the closed circuit is the same, provided the change in the magnetic field in its immediate neighbourhood is the same. In accordance with this and the previous statements, the induced electric force in a conductor in motion is equal to the number of lines of force which are cut by the conductor in a definite direction per unit of time, multiplied by A. The product of this electric force and of the current in the moving conductor gives, according to section (11), the thermal or chemical work done by induction in the conductor. It follows from the results of the preceding section, if we pay due regard to sign, that this is equal to the mechanical work which must be done by the external forces acting upon the circuit. Hence, if a current of constant strength is maintained in a circuit, and this circuit is moved towards a fixed magnet, the thermal and chemical energy developed in the circuit accounts for the mechanical work done; while the magnetic energy of the system remains constant. But, on the other hand, if this circuit is moved towards another in which a constant current is maintained, the larger amount of thermal and chemical energy developed in the one through the motion accounts for the mechanical work done; and the same extra amount of energy which appears in the other circuit accounts for the diminution in the magnetic energy of the field. Or, to speak more accurately, the sum of the former amounts of energy balances the sum of the latter. This settles the point referred to at the end of section (16).

## 18. *Electromagnetics of Unclosed Currents*

With regard to the phenomena which are possible, this is the richest region of all; for it includes all those problems which we cannot apportion elsewhere as special cases. But as far as actual experience is concerned, it is a region which hitherto has been but slightly explored. The oscillations of unclosed induction-circuits, or of discharging Leyden jars, can be treated with sufficient approximation according to the laws of the preceding section; and so far only the electric waves and oscillations of short wave-length, which have been discussed in the earlier papers, strictly belong here. With regard to the theoretical treatment of this section we must therefore refer to these earlier papers—pointing out, however, that the splitting up of the electric force into an electrostatic and an electromagnetic part does not in these general problems convey any physical meaning which can be clearly conceived, nor is it of any great mathematical use; so that, instead of following earlier methods of treatment, it will be expedient to avoid it.

## 19. *Optical Phenomena in Isotropic Bodies*

We include in optics those electromagnetic disturbances which are purely periodic in time, and whose period does not exceed a very small fraction, say the billionth ($10^{-12}$) part, of a second. By none of the means which are at our disposal for detecting such disturbances can we recognise the magnetic and electric forces as such; what we are able to detect are simply the geometrical relations according to which the existing disturbance is propagated in different directions with different intensities. Hence the mathematical representation of the phenomena may also be confined to following the propagation of one of the two kinds of force, after eliminating the opposite kind; and it is immaterial which of the two is chosen for consideration. If we restrict ourselves to homogeneous isotropic non-conductors and eliminate in the one case the electric, in the other the magnetic, force-components, we obtain from equations ($4_a$) and ($4_b$) the following equations:—

$$(19_a)\quad \begin{cases} A^2\epsilon\mu\dfrac{d^2L}{dt^2} = \Delta L, \\ A^2\epsilon\mu\dfrac{d^2M}{dt^2} = \Delta M, \\ A^2\epsilon\mu\dfrac{d^2N}{dt^2} = \Delta N, \\ \dfrac{dL}{dx} + \dfrac{dM}{dy} + \dfrac{dN}{dz} = 0, \end{cases} \qquad (19_b)\quad \begin{cases} A^2\epsilon\mu\dfrac{d^2X}{dt^2} = \Delta X, \\ A^2\epsilon\mu\dfrac{d^2Y}{dt^2} = \Delta Y, \\ A^2\epsilon\mu\dfrac{d^2Z}{dt^2} = \Delta Z, \\ \dfrac{dX}{dx} + \dfrac{dY}{dy} + \dfrac{dZ}{dz} = 0. \end{cases}$$

The solutions of these, assuming that the disturbances are purely periodic, are always solutions of the equations ($4_a$) and ($4_b$) as well. From each of the two systems of equations ($19_a$) and ($19_b$) it can be seen that transverse waves are possible, and that longitudinal waves are impossible; each of the two systems gives for the velocity of the possible waves the value

$$1/A\sqrt{\epsilon\mu};$$

from each of the two systems the phenomena of rectilinear propagation, of diffraction, of the interference of natural and of polarised light can be deduced, and the different kinds of polarisation can be understood. By returning to equations ($4_a$) and ($4_b$) it can be shown that the simultaneous directions of the electric and the magnetic force at any point of a plane wave are invariably perpendicular to one another.

Suppose that the surface of separation of two homogeneous isotropic bodies coincides with the *xy*-plane. In accordance with section (8), and bearing in mind that we are dealing only with periodic disturbances, the following conditions obtain at this surface of separation

$$(19_c)\begin{cases} L_1 = L_2, \\ M_1 = M_2, \\ \mu_1 N_1 = \mu_2 N_2; \end{cases} \qquad (19_d)\begin{cases} X_1 = X_2, \\ Y_1 = Y_2, \\ \epsilon_1 Z_1 = \epsilon_2 Z_2. \end{cases}$$

Each of these systems of limiting equations, together with the corresponding equations for the interior of both bodies, gives the laws of reflection, of refraction, of total reflection,— in fact, the fundamental laws of geometrical optics. From each of them it follows that the intensity of reflected and refracted waves is dependent upon the nature of their polarisation, and that this dependence, as well as the retardation of phase of the totally reflected waves, is in accordance with Fresnel's formulæ. If we deduce these formulæ from the equations of the electric forces ($19_b$) and ($19_d$), it will be found that the method of development corresponds with the method of deducing these formulæ as given by Fresnel himself. If we start from the equations of the magnetic force ($19_a$) and ($19_c$), we approach the path by which F. Neumann arrived at Fresnel's equations. From our more general standpoint we cannot only see from the start that both paths must lead to the same goal, but we can also recognise that the two are equally satisfactory. In the actually observed phenomena of reflection the electric and magnetic forces are not completely interchangeable, and the two paths appear to be different. This is because the magnetic permeabilities are almost the same and equal to unity for all bodies which come under consideration, whereas the specific inductive capacities differ considerably; and hence the optical behaviour of bodies is chiefly determined by their electrical properties.

If the $xy$-plane forms the boundary between our non-conductor and a perfect conductor, the following equations obtain in this plane:—

$$(19_e) \qquad N = 0,$$
$$(19_f) \qquad X = 0, \quad Y = 0.$$

From these, together with the corresponding equations for the interior of the non-conductor, it follows that for every angle of incidence and every azimuth of polarisation the reflection is total. Since all actual conductors occupy an intermediate position between perfect conductors and non-conductors, the reflection from them may be expected to be of a kind intermediate between total reflection and the reflection from transparent bodies. Inasmuch as metallic reflection occupies such a position, our equations appear adapted for giving a general picture of metallic reflection as well. Up to the present, however, investigation does not enable us to state how far such a representation, by suitable choice of the constants, can be extended into details.

It has already been pointed out in the first section that the phenomena of dispersion require the introduction of at least two electric or two magnetic quantities, and that they therefore lie outside the limits of our present theory.

### 20. *Optics of Crystalline Bodies*

We shall confine our attention to optical phenomena in the interior of a homogeneous, completely transparent crystal,—in which we further assume that the axes of symmetry of the electric and the magnetic energy coincide. Let the co-ordinate axes be parallel to these common axes of symmetry, and, for the sake of simplicity, let us write

$$\epsilon_1,\ \epsilon_2,\ \epsilon_3,\ \mu_1,\ \mu_2,\ \mu_3,\ \text{instead of}\ \epsilon_{11},\ \epsilon_{22},\ \epsilon_{33},\ \mu_{11},\ \mu_{22},\ \mu_{33}.$$

Equations ($5_a$) and ($5_b$), which here come under consideration, now take the form

$$(20_a)\begin{cases} A\mu_1\dfrac{dL}{dt} = \dfrac{dZ}{dy} - \dfrac{dY}{dz}, \\ A\mu_2\dfrac{dM}{dt} = \dfrac{dX}{dz} - \dfrac{dZ}{dx}, \\ A\mu_3\dfrac{dN}{dt} = \dfrac{dY}{dx} - \dfrac{dX}{dy}; \end{cases} \qquad (20_b)\begin{cases} A\epsilon_1\dfrac{dX}{dt} = \dfrac{dM}{dz} - \dfrac{dN}{dy}, \\ A\epsilon_2\dfrac{dY}{dt} = \dfrac{dN}{dx} - \dfrac{dL}{dz}, \\ A\epsilon_3\dfrac{dZ}{dt} = \dfrac{dL}{dy} - \dfrac{dM}{dx}. \end{cases}$$

These equations are integrated by assuming that the light consists of plane waves of plane-polarised light, corresponding to the following statements:—The magnetic force is perpendicular to the electric polarisation, and the electric force is perpendicular to the magnetic polarisation. In general the direction of both forces does not coincide with the wave-plane; the direction of both polarisations lies in the wave-plane. Hence the direction which is perpendicular to both polarisations is the wave-normal; the direction which is perpendicular to both forces is the direction in which, according to section (11), the energy is propagated; in optics it is called the ray. To every given position of the wave-normal there correspond in general two possible waves of different polarisations, different velocities, and different positions of the corresponding rays. If we suppose that at any given instant plane waves starting from the origin of co-ordinates proceed outwards in all possible directions of the wave-normals, these wave-planes after unit time envelop a surface,—the so-called wave-surface. Each single wave-plane touches the wave-surface at a point on the corresponding ray from the origin. The equation to the surface enveloped by the wave-planes is found to be

$$(20_c)\begin{cases} \left(\dfrac{x^2}{\epsilon_1} + \dfrac{y^2}{\epsilon_2} + \dfrac{z^2}{\epsilon_3}\right)\left(\dfrac{x^2}{\mu_1} + \dfrac{y^2}{\mu_2} + \dfrac{z^2}{\mu_3}\right) - \dfrac{x^2}{\epsilon_1\mu_1}\left(\dfrac{1}{\epsilon_2\mu_3} + \dfrac{1}{\epsilon_2\mu_2}\right) \\ - \dfrac{y^2}{\epsilon_2\mu_2}\left(\dfrac{1}{\epsilon_1\mu_3} + \dfrac{1}{\epsilon_3\mu_1}\right) - \dfrac{z^2}{\epsilon_3\mu_3}\left(\dfrac{1}{\epsilon_1\mu_2} + \dfrac{1}{\epsilon_2\mu_1}\right) + \dfrac{1}{\epsilon_1\epsilon_2\epsilon_3\mu_1\mu_2\mu_3} = 0. \end{cases}$$

The surface of the fourth degree represented by this equation cuts each of the co-ordinate planes in two ellipses. In one of the co-ordinate planes the two ellipses intersect each other in four points—the four conical points (*Nabelpunkte*) of the surface; in the two other co-ordinate planes one of the ellipses surrounds the other; and these statements hold good whatever the values of $\epsilon$ and $\mu$ are. To a very near approximation $\mu_1 = \mu_2 = \mu_3 = 1$ for all actual crystals; in this case the general form of the equation reduces to that of Fresnel's wave-surface, and of the two ellipses in which the surface cuts the co-ordinate planes, one reduces to a circle.

It is well known that the explanation of double refraction, of reflection at crystalline surfaces, and many of the interference-phenomena observed in crystals are intimately connected with the consideration of the wave-surface and the simpler forms which it assumes in special cases. But other facts, again, in crystallographic optics cannot be mastered by following out the idea of a single electric and a single magnetic directed magnitude; hence these facts lie outside the present limits of our theory.

In sections (17) to (20) we have completed the enumeration of those cases of the variable state whose importance has up to the present time given rise to the development of special theories.