IX

THE FORCES OF ELECTRIC OSCILLATIONS, TREATED ACCORDING TO MAXWELL'S THEORY

(*Wiedemann's Ann.* **36**, p. 1, 1889.)

THE results of the experiments on rapid electric oscillations which I have carried out appear to me to confer upon Maxwell's theory a position of superiority to all others. Nevertheless, I based my first interpretation of these experiments upon the older views, seeking partly to explain the phenomena as resulting from the co-operation of electrostatic and electromagnetic forces. To Maxwell's theory in its pure development such a distinction is foreign. Hence I now wish to show that the phenomena can be explained in terms of Maxwell's theory without introducing this distinction. Should this attempt succeed, it will at the same time settle any question as to a separate propagation of electrostatic force, which indeed is meaningless in Maxwell's theory.

Apart from this special aim, a closer insight into the play of the forces which accompany a rectilinear oscillation is not without interest.

The Formulæ

In what follows we are almost solely concerned with the forces in free ether. In this let X, Y, Z be the components of the electric force along the co-ordinates of x, y, z;[1] L, M, N,

[1] Suppose that you are standing at the origin of the system of co-ordinates on the xy-plane. Further assume that the direction of positive x is straight in front, of positive z upwards, and of positive y to the right hand. Unless these conventions were made, the signs of the electric and magnetic forces in the subsequent equations would not have their usual meanings.

the corresponding components of the magnetic force, both being measured in Gauss units;[1] and let t denote the time and A the reciprocal of the velocity of light. Then, according to Maxwell, the time-rate of change of the forces is dependent upon their distribution in space as indicated by the following equations:—

$$(1)\begin{cases} A\dfrac{dL}{dt} = \dfrac{dZ}{dy} - \dfrac{dY}{dz}, \\ A\dfrac{dM}{dt} = \dfrac{dX}{dz} - \dfrac{dZ}{dx}, \\ A\dfrac{dN}{dt} = \dfrac{dY}{dx} - \dfrac{dX}{dy}, \end{cases} \qquad (2)\begin{cases} A\dfrac{dX}{dt} = \dfrac{dM}{dz} - \dfrac{dN}{dy}, \\ A\dfrac{dY}{dt} = \dfrac{dN}{dx} - \dfrac{dL}{dz}, \\ A\dfrac{dZ}{dt} = \dfrac{dL}{dy} - \dfrac{dM}{dx}. \end{cases}$$

Originally, and therefore always, the following conditions must be satisfied:—

$$(3)\ \frac{dL}{dx} + \frac{dM}{dy} + \frac{dN}{dz} = 0, \text{ and } \frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 0.$$

The electric energy contained in a volume-element τ of the ether is equal to $\frac{1}{8\pi}\int(X^2 + Y^2 + Z^2)d\tau$; the magnetic energy is equal to $\frac{1}{8\pi}\int(L^2 + M^2 + N^2)d\tau$, the integration extending through the volume τ. The total energy is the sum of both these portions.

These statements form, as far as the ether is concerned, the essential parts of Maxwell's theory. Maxwell arrived at them by starting with the idea of action-at-a-distance and attributing to the ether the properties of a highly polarisable dielectric medium. We can also arrive at them in other ways. But in no way can a direct proof of these equations be deduced from experience. It appears most logical, therefore, to regard them independently of the way in which they have been arrived at, to consider them as hypothetical assumptions, and to let their probability depend upon the very large number of natural laws which they embrace. If we take up

[1] H. v. Helmholtz, *Wied. Ann.* **17**, p. 48, 1882.

this point of view we can dispense with a number of auxiliary ideas which render the understanding of Maxwell's theory more difficult, partly for no other reason than that they really possess no meaning,[1] if we finally exclude the notion of direct action-at-a-distance.

Multiply equations (1) by L, M, N, and equations (2) by X, Y, Z; add the equations together and integrate over a volume of which $d\tau$ is the volume-element and $d\omega$ the surface-element. We thus get—

$$\frac{d}{dt}\left\{\frac{1}{8\pi}\int(X^2+Y^2+Z^2)d\tau+\frac{1}{8\pi}\int(L^2+M^2+N^2)d\tau\right\}$$

$$=\frac{1}{4\pi A}\int\left\{NY-MZ)\cos n,x+(LZ-NX)\cos n,y\right.$$

$$\left.+(MX-LY)\cos n,z\right\}d\omega,$$

where n,x n,y n,z denote the angles which the normals from $d\omega$ make with the axis.

This equation shows that the amount by which the energy of the space has increased can be regarded as having entered through the elements of the surface. The amount which enters through each element of the surface is equal to the product of the components of the electric and magnetic forces resolved along the surface, multiplied by the sine of the angle which they form with each other, and divided by $4\pi A$. It is well known that upon this result Dr. Poynting has based a highly remarkable theory on the transfer of energy in the electromagnetic field.[2]

With regard to the solution of the equations we restrict ourselves to the special but important case in which the distribution of the electric force is symmetrical about the z-axis, in such a way that this force at every point lies in the meridian plane passing through the axis of z and only depends upon the z-co-ordinate of the point and its distance $\rho=\sqrt{x^2+y^2}$ from the z-axis. Let R denote the component of the electric force in the direction of ρ, namely $Xx/\rho+Yy/\rho$; and further let P denote the component of the magnetic force

[1] As an example I would mention the idea of a dielectric-constant of the ether.

[2] J. H. Poynting, *Phil. Trans.*, 1884, II. p. 343.

perpendicular to the meridian plane, namely $Ly/\rho - Mx/\rho$. We then assert that if Π is any function whatever of ρ, z, t, which satisfies the equation—

$$A^2 d^2\Pi/dt^2 = \Delta\Pi,$$

and if we put $Q = \rho d\Pi/d\rho$, then the system

$$\rho Z = dQ/d\rho, \qquad \rho P = A dQ/dt,$$
$$\rho R = -dQ/dz, \qquad N = 0$$

is a possible solution of our equations.

In order to prove this assertion, we observe that we have—

$$X = R\frac{d\rho}{dx} = -\frac{d^2\Pi}{dx\,dz}, \qquad L = P\frac{d\rho}{dy} = A\frac{d^2\Pi}{dy\,dt},$$

$$Y = R\frac{d\rho}{dy} = -\frac{d^2\Pi}{dy\,dz}, \qquad M = -P\frac{d\rho}{dx} = -A\frac{d^2\Pi}{dx\,dt},$$

$$Z = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\Pi}{d\rho}\right) = \frac{d^2\Pi}{dx^2} + \frac{d^2\Pi}{dy^2}, \qquad N = 0.$$

We have only to substitute these expressions in the equations (1), (2) and (3) to find equations (2) and (3) identically satisfied, and also equations (1) if we have regard to the differential equation for Π.

It may also be mentioned that conversely, neglecting certain limitations of no practical importance, every possible distribution of electric force which is symmetrical about the z-axis can be represented in the above form; but for the purpose of what follows it is not necessary to accept this statement.

The function Q is of importance to us. For the lines in which the surface of revolution $Q = \text{const.}$ cuts its meridian planes are the lines of electric force; if we construct these for every meridian plane at any instant we get a clear representation of the distribution of the force. If we cut the cup-shaped space lying between the surfaces Q and $Q + dQ$ in various places by surfaces of rotation around the z-axis, then for all such cross-sections the product of electric force and cross-section, which Maxwell calls the induction across the section, is the same. If we arrange the system of surfaces $Q = \text{const.}$ so that in passing from one to another Q increases by the same amount dQ, then the same statement holds good if we compare amongst themselves the cross-sections of the various spaces thus formed. In the plane diagram formed by the intersection of the meridian planes with the equidistant surfaces $Q = \text{const.}$, the electric force is only inversely proportional to the perpendicular distance between two of the lines $Q = \text{const.}$ when the points compared lie at the same distance from the z-axis; in general, the rule is that the force is inversely proportional to the product of this distance, and of the co-ordinate ρ of the point under consideration.

In what follows we shall introduce along with ρ and z the polar co-ordinates r and θ, which are connected with the former by the relations $\rho = r \sin \theta$, $z = r \cos \theta$; r then denotes the distance from the origin of our system of co-ordinates.

The Forces around a Rectilinear Oscillation

Let E denote a quantity of electricity, l a length, $m = \pi/\lambda$ the reciprocal of a length, and $n = \pi/T$ the reciprocal of a time. Let us put

$$\Pi = El \frac{\sin (mr - nt)}{r}.$$

This value satisfies the equation $A^2 d^2\Pi/dt^2 = \Delta\Pi$, if we stipulate that $m/n = T/\lambda = A$, and hence that λ/T shall be equal to the velocity of light. And it must be noticed that the equation referred to is satisfied everywhere, except at the origin of our system of co-ordinates.

In order to find out what electrical processes at this point correspond to the distribution of force specified by Π, let us investigate its immediate neighbourhood. Thus let r be vanishingly small compared with λ, and mr negligible compared with nt. Then Π becomes[1] equal to $-El \sin nt/r$. Now since

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right) \left(\frac{1}{r}\right) = -\frac{d^2}{dz^2}\left(\frac{1}{r}\right),$$

we have

$$X = -d^2\Pi/dxdz, \quad Y = -d^2\Pi/dydz, \quad Z = -d^2\Pi/dzdz.$$

[1] [See Note 22 at end of book.]

Thus the electric forces appear here as the derivatives of a potential—

$$\phi = \frac{d\Pi}{dz} = -\mathrm{E}l \sin nt \frac{d}{dz}\left(\frac{1}{r}\right),$$

and this corresponds to an electrical double-point, whose axis coincides with the z-axis, and whose moment oscillates between the extreme values $+\mathrm{E}l$ and $-\mathrm{E}l$ with the period T. Hence our distribution of force represents the action of a rectilinear oscillation which has the very small length l, and on whose poles at the maximum the quantities of electricity $+$ E and $-$ E become free. The magnetic force perpendicular to the direction of the oscillation and in its immediate neighbourhood comes out as—

$$\mathrm{P} = \mathrm{AE}ln \cos nt \sin \theta/r^2.$$

According to the Biot-Savart law this is the force of a current-element of length l lying in the direction of the axis of z, and whose intensity, in magnetic measure, oscillates between the extreme values $+\pi\mathrm{AE/T}$ and $-\pi\mathrm{AE/T}$. In fact, the motion of the quantity of electricity E determines a current of that magnitude.

From Π we get—

$$\mathrm{Q} = \mathrm{E}lm\left\{\cos(mr - nt) - \frac{\sin(mr - nt)}{mr}\right\}\sin^2\theta,$$

and from this the forces Z, R, P follow by differentiation. Now it is true that the formulæ in general turn out to be too complicated to allow of a direct survey of the distribution of the forces. But in some special cases, which we will now indicate, the results are comparatively simple—

(1) We have already considered the immediate neighbourhood of the oscillation.

(2) In the z-axis, *i.e.* in the direction of the oscillation, we have $d\rho = rd\theta$, $dz = dr$, $\theta = 0$; so that here

$$\mathrm{R} = 0, \qquad \mathrm{P} = 0,$$

$$\mathrm{Z} = \frac{2\mathrm{E}lm}{r^2}\left\{\cos(mr - nt) - \frac{\sin(mr - nt)}{mr}\right\}.$$

The electric force acts always in the direction of the oscillation; at small distances it diminishes as the inverse cube, at greater distances as the inverse square, of the distance.

(3) In the xy-plane, *i.e.* when $z=0$, we have $dz = -rd\theta$, $d\rho = dr$, $\theta = 90^0$; and therefore—

$$P = \frac{AElmn}{r}\left\{\sin(mr-nt) + \frac{\cos(mr-nt)}{mr}\right\},$$
$$R = 0,$$
$$Z = \frac{Elm^2}{r}\left\{-\sin(mr-nt) - \frac{\cos(mr-nt)}{mr} + \frac{\sin(mr-nt)}{m^2r^2}\right\}.$$

In the equatorial plane through the oscillation the electric force is parallel to the oscillation, and its amplitude is $El\sqrt{1-m^2r^2+m^4r^4}/r^3$. The force diminishes continuously with increasing distance, at first rapidly as the inverse cube, but afterwards only very slowly and inversely as the distance itself. At greater distances the action of the oscillation can only be observed in the equatorial plane, and not along the axis.

(4) At very great distances we may neglect higher powers of $1/r$ as compared with lower ones. Thus we have at such distances—

$$Q = Elm\cos(mr-nt)\sin^2\theta,$$

from which we deduce—

$$P = A\,.\,Elmn\sin(mr-nt)\sin\theta/r,$$
$$Z = -Elm^2\sin(mr-nt)\sin^2\theta/r,$$
$$R = Elm^2\sin(mr-nt)\sin\theta\cos\theta/r.$$

Whence it follows that $Z\cos\theta + R\sin\theta = 0$. Hence at great distances the force is everywhere perpendicular to the radius vector from the origin of the force; the propagation takes place in the form of a pure transversal wave. The magnitude of the force is $Elm^2\sin(mr-nt)\sin\theta/r$. At a constant distance from the zero-point it decreases towards the axis, being proportional to the distance from the latter.

In order now to find the distribution of force in the remaining parts of space we make use of graphic representation, drawing for definite times the lines of electric force, *i.e.* the curves $Q = \text{const.}$, for equidistant values of Q. Since Q appears as the product of two factors, of which the one depends only upon r, and the other only upon θ, the construction of these curves presents no great difficulty. We

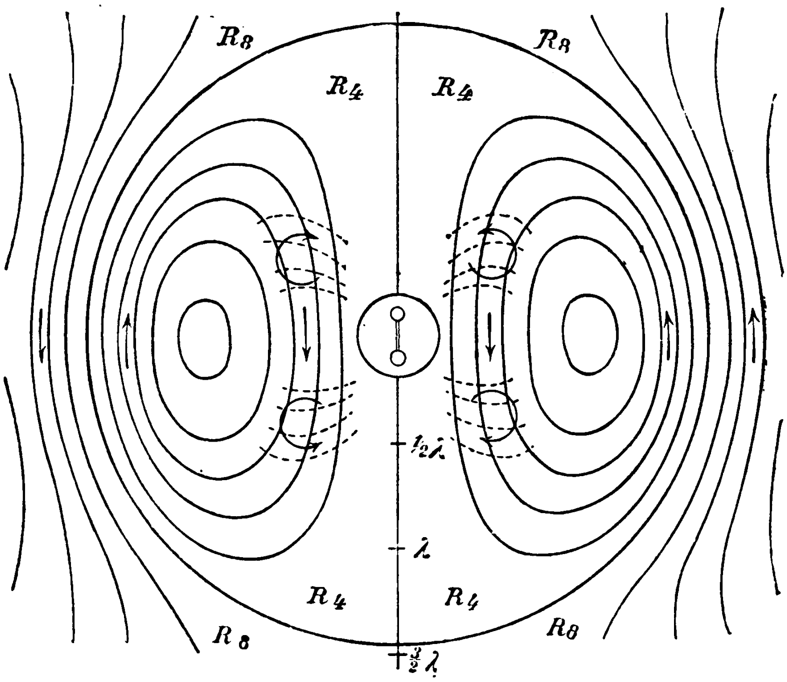

Fig. 27.

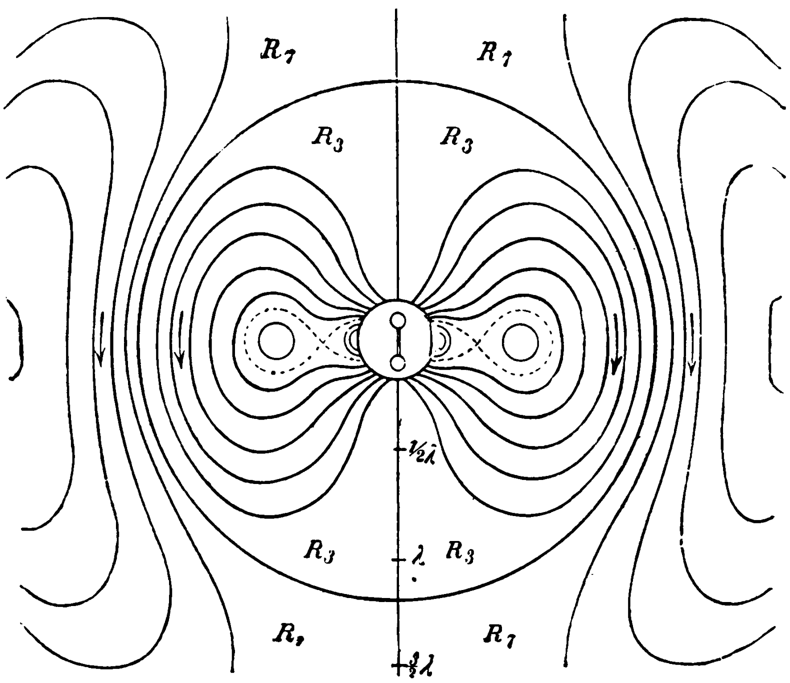

Fig. 30.

split up each value of Q, for which we wish to draw the curve, in various ways into two factors; we determine the angle θ for which $\sin^2\theta$ is equal to the one factor and, by means of an auxiliary curve, the value of r for which the function of r contained in Q is equal to the other factor; in

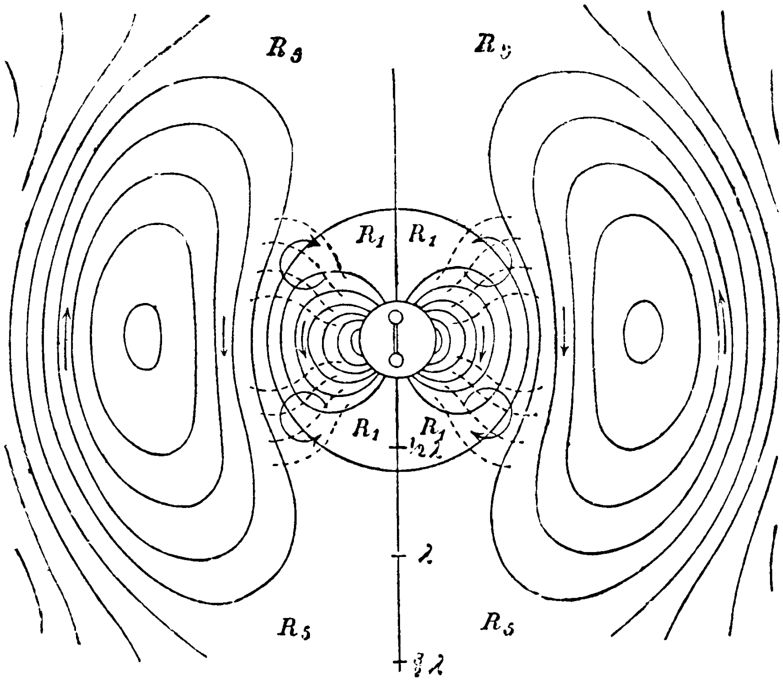

Fig. 28.

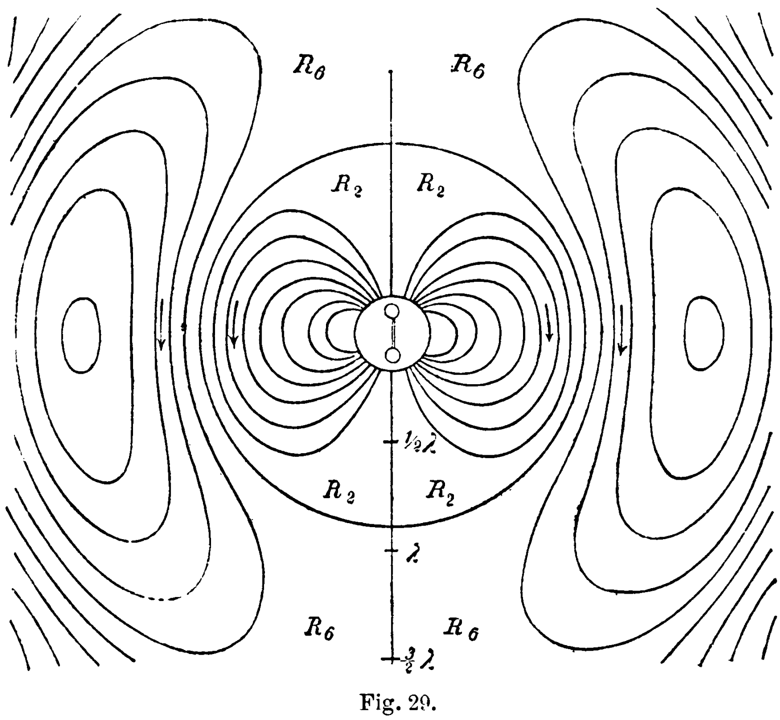

Fig. 29.

this way we find as many points on the curve as we please. On setting about the construction of these curves one perceives many small artifices which it would be tedious to exhibit here.

We shall content ourselves with considering the results of the construction as shown in Figs. 27-30. These figures exhibit the distribution of force at the times $t = 0$, $\frac{1}{4}$T, $\frac{1}{2}$T, $\frac{3}{4}$T, or by a suitable reversal of the arrows for all subsequent times which are whole multiples of $\frac{1}{4}$T. At the origin is shown, in its correct position and approximately to correct scale, the arrangement which was used in our earlier experiments for exciting the oscillations. The lines of force are not continued right up to this picture, for our formulæ assume that the oscillator is infinitely short, and therefore become inadequate in the neighbourhood of the finite oscillator.

Let us begin our explanation of the diagrams with Fig. 27. Here $t = 0$; the current is at its maximum strength, but the poles of the rectilinear oscillator are not charged with electricity—no lines of force converge towards them. But from the time $t = 0$ onwards, such lines of force begin to shoot out from the poles; they are comprised within a sphere represented by the value Q = 0. In Fig. 27, indeed, this sphere is still vanishingly small, but it rapidly enlarges, and by the time $t = \frac{1}{4}$T (Fig. 28) it already fills the space R_1. The distribution of the lines of force within the sphere is nearly of the same kind as that corresponding to a static electric charge upon the poles. The velocity with which the spherical surface Q = 0 spreads out from the origin is at first much greater than 1/A; in fact, for the time $\frac{1}{4}$T this latter velocity would only correspond to the value of $\frac{1}{4}\lambda$ given in the figure. At an infinitesimal distance from the origin the velocity of propagation is even infinite. This is the phenomenon which, according to the old mode of expression, is represented by the statement that upon the electromagnetic action which travels with the velocity 1/A, there is superposed an electrostatic force travelling with infinite velocity. In the sense of our theory we more correctly represent the phenomenon by saying that fundamentally the waves which are being developed do not owe their formation solely to processes at the origin, but arise out of the conditions of the whole surrounding space, which latter, according to our theory, is the true seat of the energy. However this may be, the surface Q = 0 spreads out further with a velocity which gradually sinks to 1/A, and by the time $t = \frac{1}{2}$T (Fig. 29) fills the space R_2. At this time the electrostatic charge of the poles is at its greatest development; the number of lines of force which converge towards the poles is a maximum. As time progresses further no fresh lines of force proceed from the poles, but the existing ones rather begin to retreat towards the oscillating conductor, to disappear there as lines of electric force, but converting their energy into magnetic energy. Here there arises a peculiar action which can plainly be recognised, at any rate in its beginnings, in Fig. 30 ($t = \frac{3}{4}$T). The lines of force which have withdrawn furthest from the origin become laterally inflected by reason of their tendency to contract together; as this inflection contracts nearer and nearer towards the z-axis, a portion of each of the outer lines of force detaches itself as a self-closed line of force which advances independently into space, while the remainder of the lines of force sink back into the oscillating conductor.

The number of receding lines of force is just as great as the number which proceeded outwards, but their energy is necessarily diminished by the energy of the parts detached. This loss of energy corresponds to the radiation into space. In consequence of this the oscillation would of necessity soon come to rest unless impressed forces restored the lost energy at the origin. In treating the oscillation as undamped, we have tacitly assumed the presence of such forces. In Fig. 27 —to which we now return for the time $t = $ T, conceiving the arrows to be reversed—the detached portions of the lines of force fill the spherical space R_4, while the lines of force proceeding from the poles have completely disappeared. But new lines of force burst out from the poles and crowd together the lines whose development we have followed into the space R_5 (Fig. 28). It is not necessary to explain further how these lines of force make their way to the spaces R_6 (Fig. 29), R_7 (Fig. 30), R_8 (Fig. 27). They run more and more into a pure transverse wave-motion, and as such lose themselves in the distance. The best way of picturing the play of the forces would be by making drawings for still shorter intervals of time and attaching these to a stroboscopic disk.

A closer examination of the diagrams shows that at points which do not lie either on the z-axis or in the xy-plane the direction of the force changes from instant to instant. Thus, if we represent the force at such a point in the usual manner by a line drawn from the point, the end of this line will not simply move backwards and forwards along a straight line during an oscillation, but will describe an ellipse. In order to find out whether there are any points at which this ellipse approximates to a circle, and in which, therefore, the force turns successively through all points of the compass without any appreciable change of magnitude, we superpose two of the diagrams which correspond to times differing by $\frac{1}{2}$T from one another, *e.g.* Figs. 27 and 29, or 28 and 30. At such points as we are trying to find, the lines of the one system must clearly cut those of the second system at right angles, and the distances between the lines of the one system must be equal to those of the second. The small quadrilaterals formed by the intersection of both systems must therefore be squares at the points sought. Now, in fact, regions of this kind can be observed; in Figs. 27 and 28 they are indicated by circular arrows, the directions of which at the same time give the direction of rotation of the force. For further explanation dotted lines are introduced which belong to the system of lines in Figs. 29 and 30. Furthermore, we find that the behaviour here sketched is exhibited by the force not only at the points referred to, but also in the whole strip-shaped tract which, spreading out from these points, forms the neighbourhood of the *z*-axis. Yet the force diminishes in magnitude so rapidly in this direction that its peculiar behaviour only attracts attention at the points mentioned.

In an imperfect series of observations which are not guided by theory, the force-system here described, and required by theory, may well exhibit itself in the manner described in an earlier paper.[1] The observations referred to do not by any means enable us to recognise all the complicated details, but they show correctly the main features of the distribution. According to both observation and theory the distribution of the force in the neighbourhood of the oscillator is similar to the electrostatic distribution. According to both observation and theory the force spreads out chiefly in the equatorial plane and diminishes in that plane at first rapidly, then slowly, without becoming zero

[1] See V., p. 90.

at any intermediate distance. According to both observation and theory the force in the equatorial plane, along the axis, and at great distances, is constant in direction and variable in magnitude; whereas, at intermediate points, its direction varies greatly and its magnitude but little. The only want of accord between theory and the observations referred is in this—that, according to the former, the force at great distances should always be perpendicular to the radius vector from the origin, whereas, according to the latter, it appeared to be parallel to the oscillation. These two come to the same thing for the neighbourhood of the equatorial plane, where the forces are strongest, but not for directions lying between the equatorial plane and the axis. I believe that the error is on the side of the observations. In the experiments referred to the oscillator was parallel to the two main walls of the room used; and the components of the force parallel to the oscillator might thereby be strengthened as compared with the components normal to the oscillator.

I have therefore repeated the experiments, making various alterations in the position of the primary oscillator, and found that in certain positions the results were in accordance with theory. Nevertheless, the results were not free from ambiguity, for at great distances and in places where the force was feeble, the disturbances due to the environment of the space at my disposal were so considerable that I could not arrive at a trustworthy decision.

While the oscillator is at work the energy oscillates in and out through the spherical surfaces surrounding the origin. But the energy which goes out during each period of oscillation through every surface is greater than that which returns, and is greater by the same amount for all the surfaces. This excess represents the loss of energy due to radiation during each period of oscillation. We can easily calculate it for a spherical surface, whose radius r is so great that we may use the simplified formulæ. Thus the energy which goes out in the element of time dt through a spherical zone lying between θ and $\theta + d\theta$ is

$$dt \,.\, 2\pi r \sin\theta \,.\, r d\theta \,.\, (\mathrm{Z} \sin\theta - \mathrm{R} \cos\theta)\, \mathrm{P} \,.\, 1/4\pi \mathrm{A}.$$

If we here substitute for Z, P, R, the values corresponding to large values of r and integrate with respect to θ from 0 to π, and with respect to t from 0 to T, we get for the energy which goes out through the whole sphere during a half-oscillation

$$\tfrac{1}{3}E^2l^2m^3nt = \pi^4E^2l^2/3\lambda^3.$$

Let us now try to deduce from this an approximate estimate of the quantities actually involved in our experiments. In these we charged two spheres of 15 cm. radius in opposite senses up to a sparking distance of about 1 cm. If we estimate the difference of potential between the two spheres as 120 C.G.S. electrostatic units ($gm.^{\frac{1}{2}}\ cm.^{\frac{1}{2}}\ sec.^{-1}$), then each sphere was charged to a potential of $\pm$ 60 C.G.S. units, and therefore its charge was $E = 15 \times 60 = 900$ C.G.S. units ($gm.^{\frac{1}{2}}\ cm.^{\frac{3}{2}}\ sec.^{-1}$). Hence the whole stock of energy which the oscillator possessed at the start amounted to $2 \times \frac{1}{2} \times 900 \times 60 = 54{,}000$ ergs, or about the energy which a gramme-weight would acquire in falling through 55 cm. The length l of the oscillator was about 100 cm., and the wave-length about 480 cm. Hence it follows that the loss of energy in the half-period of oscillation was about 2400 ergs.[1] It is therefore evident that after eleven half-oscillations one-half of the energy will have been expended in radiation. The rapid damping of the oscillations, indicated by our experiments, was therefore necessarily determined by the radiation, and would still occur even if the resistance of the conductor and of the spark became negligible.

To furnish energy amounting to 2400 ergs in 1·5 hundred-millionths of a second is equivalent to working at the rate of 22 horse-power. The primary oscillator must be supplied with energy at fully this rate if its oscillations are to be kept up continuously and with constant intensity in spite of the radiation. During the first few oscillations the intensity of the radiation at a distance of about 12 metres from the primary conductor corresponds to the intensity of the sun's radiation at the surface of the earth.

The Interference-Experiments

In order to ascertain the velocity of propagation of the electric force in the equatorial plane, we caused it to interfere

[1] [See Note 23 at end of book.]

with the action of an electric wave proceeding with constant velocity along a wire.[1] It appeared that the resulting interferences did not succeed each other at equal distances, but that the changes were more rapid in the neighbourhood of the oscillation than at greater distances. This behaviour was explained by the supposition that the total force might be split up into two parts, of which the one, the electromagnetic, was propagated with the velocity of light, while the other, the electrostatic, was propagated with a greater, and perhaps infinite velocity. But now, according to our theory, the force under consideration in the equatorial plane is—

$$Z = Elm^3\left\{ -\frac{\sin(mr - nt)}{mr} - \frac{\cos(mr - nt)}{m^2r^2} + \frac{\sin(mr - nt)}{m^3r^3} \right\},$$

and this expression can in no way be split up into two simple waves travelling with different velocities. Hence if our present theory is correct, the earlier explanation can only serve as an approximation to the truth. Let us now investigate whether the present theory leads to any explanation of the phenomena.

To begin with, we can write $Z = B \sin(nt - \delta_1)$, where the amplitude of the force $B = El\sqrt{1 - m^2r^2 + m^4r^4}/r^3$, and the phase δ_1 of the force is determined by the equation—

$$\tan \delta_1 = \frac{\sin mr/mr + \cos mr/m^2r^2 - \sin mr/m^3r^3}{\cos mr/mr - \sin mr/m^2r^2 - \cos mr/m^3r^3},$$

which, after transformation, gives

$$\delta_1 = mr - \tan^{-1}\frac{mr}{1 - m^2r^2}.$$

In Fig. 31 the quantity δ_1 is represented as a function of mr by the curve δ_1. The length ab in the figure corresponds to the value of π, both for abscissæ and ordinates. If we regard r, instead of mr, as the variable abscissa, the length ab in the abscissæ corresponds to the half wave-length. For the purpose of referring directly to the experiments which we wish to discuss, there is placed beneath the diagram a further division of the axis of abscissæ into metres. According to the results obtained by direct experiment[2] λ is put $= 4{\cdot}8$ metres, and

[1] See VII., p. 107.

[2] See VIII., p. 124.

from this the length of the metre (or scale of divisions) is determined; but the first mark of the divided scale is not at the oscillator, but is placed at a distance of 0·45 metre beyond the latter. In this way the divisions represent the divisions

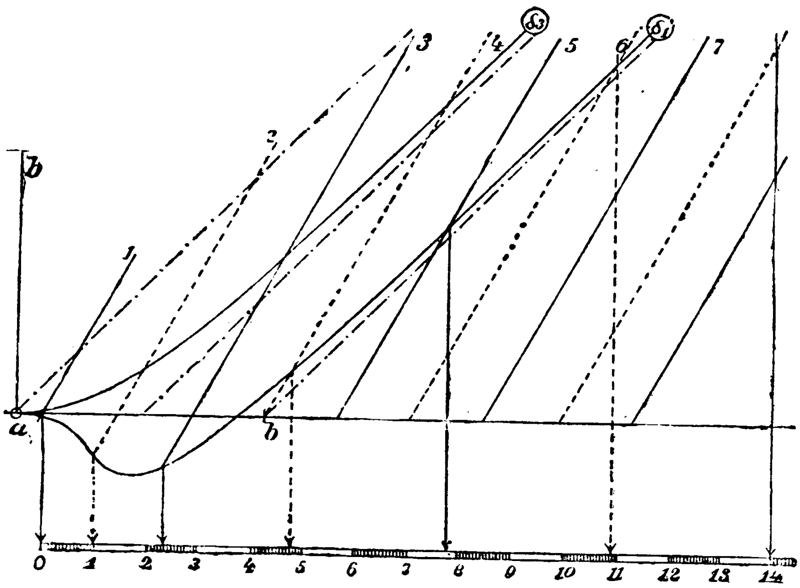

Fig. 31.

of the base-line which we used in determining the interferences. We see from the figure that the phase does not increase from the source; its course is rather as if the waves originated at a distance of about $\frac{1}{2}\lambda$ in space and spread out thence, partly towards the conductor, and partly into space. At great distances the phase is smaller by the value π than it would have been if the waves had proceeded with constant velocity from the origin; the waves, therefore, behave at great distances as if they had travelled through the first half wave-length with infinite velocity.

The action w of the waves in the wire for a given position of the secondary conductor can now be represented in the form $w = C \sin(nt - \delta_2)$, wherein δ_2 is used as an abbreviation for $m_1 r + \delta = \pi r / \lambda_1 + \delta$. λ_1 denotes the half wave-length of the waves in the wire, which in our experiments was 2·8 metres. δ indicates the phase of its action at the point $r = 0$, which we altered arbitrarily by interposing wires of various lengths. Similarly we were able to alter the amplitude C, and we made it of such magnitude that the action of the waves in the wire was approximately equal to the direct action. The phase of the interference then depends only upon the difference between the phases δ_1 and δ_2. With that particular adjustment of the secondary circle to which our expression for w relates, both actions reinforce one another (*i.e.* the interference has the sign +) if $\delta_1 - \delta_2$ is equal to zero or an odd multiple of 2π; the actions annul one another (*i.e.* the interference has the sign −) if $\delta_1 - \delta_2$ is equal to π or an odd multiple of it; no interference takes place (the interference has the sign 0) if $\delta_1 - \delta_2$ is equal to an odd multiple of $\frac{1}{2}\pi$.

Let us now suppose that δ is so determined that, at the beginning of the metre-scale, the phase of the interference has a definite value ϵ, so that $\delta_1 = \delta_2 + \epsilon$. The straight line 1 in our figure will then represent the value of $\delta_2 + \epsilon$ as a function of the distance. For the inclination of the line is so chosen that for an increase of abscissa by $\lambda_1 = 2{\cdot}8$ metres, the ordinate increases by the value π; and it is so placed that it cuts the curve δ_1 at a point whose abscissa is at the beginning of the metre-scale. The lines 2, 3, 4, etc., further represent the course of the values of $\delta_2 + \epsilon - \frac{1}{2}\pi$, $\delta_2 + \epsilon - \pi$, $\delta_2 + \epsilon - \frac{3}{2}\pi$, etc. For these lines are parallel to the line 1, and are so drawn that they cut any given ordinate at distances of $\frac{1}{2}\pi$, and any given abscissa at distances of 1·4 metre. If we now project the points of intersection of these straight lines with the curve δ_1 upon the axis of abscissæ below, we clearly obtain those distances for which δ_1 is equal to $\delta_2 + \epsilon + \frac{1}{2}\pi$, $\delta_2 + \epsilon + \pi$, $\delta_2 + \epsilon + \frac{3}{2}\pi$, etc., *i.e.* for which the phase of the interference has increased by $\frac{1}{2}\pi$, π, $\frac{3}{2}\pi$, etc., as compared with the zero-point. We thus deduce directly from the figure the following statements:—If at the zero-point of the base-line the interference has the sign + (−), it first attains the sign 0 at about 1 metre, the sign − (+) at about 2·3 metres, and it again acquires the sign 0 at about 4·8 metres: the interference reverts to the sign + (−) at about 7·6 metres, is again 0 at about 14 metres, and from there on the signs succeed each other in order at about equal distances. If at the zero-point of the base-line the interference has the sign 0, it will also have this sign at about 2·3 metres, 7·6 metres, and 14 metres; it will have a marked positive or negative character at about 1 metre, 4·8 metres, and 11 metres from the zero-point. Intermediate values correspond to intermediate phases. If this theoretical result is compared with the experimental result, and especially with those interferences which occurred on introducing 100, 250, 400, and 550 cm. of wire,[1] the accordance will be found as complete as could possibly be expected.

I have not been able to account so well for the interferences of the second kind.[2] For producing these interferences we used the secondary circle in a position in which the most important factor was the integral force of induction around the closed circle. If we regard the dimensions of the latter as vanishingly small, the integral force is proportional to the rate of change of magnetic force perpendicular to the plane of the circle, and is therefore proportional to the expression—

$$\frac{dP}{dt} = AElm^2n^2\left\{-\frac{\cos(mr - nt)}{mr} + \frac{\sin(mr - nt)}{m^2r^2}\right\}.$$

Hence we deduce the phase δ_3 of this action—

$$\tan\delta_3 = -\frac{\cos mr/mr - \sin mr/m^2r^2}{\sin mr/mr + \cos mr/m^2r^2},$$

or, after transformation—

$$\delta_3 = mr - \tan^{-1} mr.$$

The line δ_3 of Fig. 31 represents the course of this function. We see that the phase of this action increases continuously from the origin itself. Hence the phenomena which point to a finite rate of propagation must, in the case of these interferences, make themselves felt even close to the oscillator. This was indeed apparent in the experiments, and therein lay the advantage presented by this kind of interference. But, contrary to the experiment, the apparent velocity near to the oscillator comes out greater than at a distance from it; and it cannot be denied that, according to theory, the change of phase of the interference should be slightly, but noticeably, more rapid than it was in the experiments. It seems to me probable that a more complete theory—in which the two conductors used would not be regarded as vanishingly small,—and perhaps a different estimate of the value of λ, would establish a more satisfactory agreement.

[1] See p. 118. [2] See p. 119.

It is, however, important to notice that even on the basis of Maxwell's theory, the numerical results obtained cannot be explained without assuming a considerable difference between the rates of propagation of the waves in wires and in free space.

Waves in Wire-shaped Conductors

The function $$K(p\rho) = \int_0^\infty e^{-\frac{1}{2}p\rho(e^u + e^{-u})}du,$$

which, for large values of ρ, approximates asymptotically to the function $\sqrt{\pi/2p\rho} \,.\, e^{-p\rho}$, and for infinitesimal values of ρ to the function $-\log(p\rho/2) - 0{\cdot}577$, satisfies the differential equation—

$$\frac{d^2K(p\rho)}{d\rho^2} + \frac{1}{\rho}\frac{dK(p\rho)}{d\rho} - p^2K(p\rho) = 0.$$

If we therefore put—

$$\Pi = \frac{2J}{An} \,.\, \sin\,(mz - nt) \,.\, K(p\rho),$$

then Π satisfies the equation $A^2d^2\Pi/dt^2 = \Delta\Pi$, if we make $p^2 = m^2 - A^2n^2$. Here J must be understood to represent a current expressed in magnetic measure, p and $m = \pi/\lambda$ reciprocals of lengths, and $n = \pi/T$ the reciprocal of a time. The function Π satisfies its equation through all space, except along the z-axis, where it is discontinuous. The values R, Z, P, N, which can be deduced from the above Π, represent therefore an electrical disturbance taking place in a very thin wire stretched along the z-axis. In the immediate neighbourhood of this wire, neglecting quantities which contain even powers of ρ, we have—

$$Q_0 = -\frac{2J}{An} \,.\, \sin\,(mz - nt),$$

and therefore—

$$R_0 = \frac{2Jm}{An\rho} \,.\, \cos\,(mz - nt),$$

$$P_0 = \frac{2J}{\rho} \,.\, \cos\,(mz - nt),$$

in which the suffix 0 indicates that ρ is assumed to be vanishingly small. From the expression for R_0 it follows that the quantity of free electricity e in unit length of the wire is—

$$e = \frac{1}{4\pi} \cdot 2\pi\rho \cdot R_0 = \frac{Jm}{An} \cdot \cos(mz - nt).$$

Similarly from the expression for P_0 it follows that the current i is—

$$i = \frac{1}{4\pi} \cdot 2\pi\rho \cdot P_0 = J \cos(mz - nt).$$

The values of i and e satisfy of themselves the necessary equation $A de/dt = -di/dz$. They show us that the disturbance under consideration is an electric sine-wave which is propagated in the positive direction along the axis of z, whose half wave-length is λ, and half-period of oscillation T, whose velocity is therefore $\lambda/T = n/m$, and whose intensity is such that the maximum current which arises is $\pm$ J.

If we stipulate that external forces may be made to act arbitrarily in the wire, we may regard λ and T as being independent of each other. For every given relation between these quantities, *i.e.* for every given velocity of the waves, the lines of electric force have a definite form which, independently of time, glides along the wire. As before, we represent this form, drawing the lines Q = const.

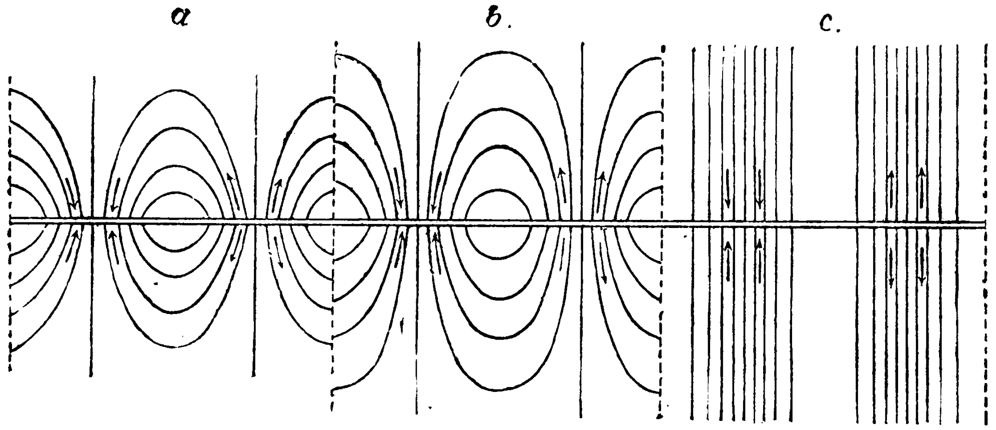

Fig. 32.

Such a representation is carried out in Fig. 32. In the first place, Fig. 32_a represents the case in which the velocity is very small and therefore $p = m$. The drawing then represents a distribution of electrostatic force, viz. that which is obtained when we distribute electricity upon the wire so that its density is a sine-function of the length of the wire. Fig. 32_b gives the lines of force for a velocity amounting to nearly 28/48 of that of light. We see that in proceeding from and returning to the wire the lines of force make a wider circuit than before. According to the older mode of conception, this would be explained by saying that the electromagnetic force, which is parallel to the wire, weakens the component of the electrostatic force in the same direction, whereas it does not affect the component perpendicular to the wire. The weakening of the component parallel to the wire may even amount to annulling it altogether. For if we take the velocity of propagation of the wire-waves as being equal to that of light, p becomes zero, $K(p\rho)$ reduces to $-\log\rho + \text{const.}$ for every value of ρ, and for every value of ρ—

$$Q = -\frac{2J}{An} \cdot \sin\,(mz - nt),$$

and therefore—

$$R = \frac{2Jm}{An\rho} \cdot \cos\,(mz - nt),\ Z = 0,$$

$$P = \frac{2J}{\rho} \cdot \cos\,(mz - nt), \quad N = 0.$$

The distribution of force then is the simplest that can be conceived; the electric force is everywhere normal to the wire and decreases in inverse proportion to the distance from it. The lines $Q = \text{const.}$, drawn for equidistant values of Q, are represented in Fig. 32_c. For waves travelling with a velocity greater than $1/A$, p becomes imaginary. For this case our formulæ would require transformation, but as it has no practical significance, we need not discuss it.

At the surface of a conductor, that component of the electric force which is tangential to the surface continues without discontinuity in the interior of the conductor. According to Maxwell, a perfect conductor is understood to mean one in whose interior there can only exist vanishingly small forces. From this it necessarily follows that at the surface of a perfect conductor the components of the force tangential to the surface must vanish. Unless this statement is incorrect, it follows that electric waves in wires of good conductivity must be propagated with the velocity of light and in the form which is represented in Fig. 32_c. For only in this particular force-distribution is the force everywhere normal to the surface of the wire. In fact, then, it follows from Maxwell's theory, as well as from the older theories, that electric waves travel along perfectly conducting wires with the velocity of light

If, on the other hand, we are to place any reliance upon our experiments, this conclusion is incorrect—the propagation takes place with a much smaller velocity and in some such form as is indicated in Fig. 32_b. The result is all the more remarkable, because the velocity in wires appears likewise to be a velocity which is quite independent of the nature of the wire. I have found it to be the same in wires of the most diverse metals, varying widely in thickness and in the shape of their cross-section, and also in columns of conducting fluids. The causes which determine this velocity still remain obscure. The resistance, at all events, has nothing to do with it. For some time I thought that it might be affected by the constant k, through the introduction of which Hr. H. v. Helmholtz has extended Maxwell's theory;[1] but further consideration led to the rejection of this idea. If only the limiting condition were correct, a wave of the form of Fig. 32_c would yet be possible. This would always be a pure transversal wave, and as such must travel with the same velocity as plane transversal waves in space, whether simultaneous longitudinal waves are possible or not. Although a finite value of the constant k would not explain the difference between the two velocities, it would postulate the possibility of two kinds of waves in the wire with different velocities: experiment has hitherto given no intimation of such a phenomenon. It seems rather to be doubtful whether the limiting condition is correct for rapidly alternating forces.

Although it does not appear to be possible, on the one hand, to confer a velocity of any desired magnitude upon the waves travelling along the z-axis, there is no difficulty, on the other hand, in reducing the velocity as much as may be desired below its maximum value, or in producing distributions of force intermediate between the forms 32_a and 32_b. With this object the waves are made to proceed along crooked wires or wires rolled into spirals. For example, I rolled a wire 40 metres

[1] H. v. Helmholtz, *Ges. Abh.* 1, p. 545.

long into a spiral 1 cm. in diameter, and so tightly that the length of the spiral was 1·6 metre; in this I was able to observe nodes at distances of about 0·31 metre, whereas, in the straight wire, the nodes were 2·8 metres apart. As the spiral was stretched out, the one value changed gradually into the other. Hence, when the velocity is measured along the z-axis (the axis of the spiral), the wave moves much more slowly in the coiled wire. When the velocity is measured along the wire itself, on the other hand, the wave certainly moves more rapidly. Along crooked wires the behaviour is similar. Unless I am mistaken, Maxwell's theory, assuming the limiting condition for good conductors, is unable to account for this. It seems to me that according to this theory the propagation, measured along the z-axis, must for every form of conductor take place with the velocity of light; provided, in the first place, that the resistance of the conductor does not come into consideration, and, in the second place, that the dimensions of the conductor perpendicular to the axis are negligible in comparison with the wave-length. Both conditions are satisfied by coiled metallic wires; but what should happen does not happen.

In our endeavour to explain the observations by means of Maxwell's theory, we have not succeeded in removing all difficulties. Nevertheless, the theory has been found to account most satisfactorily for the majority of the phenomena; and it will be acknowledged that this is no mean performance. But if we try to adapt any of the older theories to the phenomena, we meet with inconsistencies from the very start, unless we reconcile these theories with Maxwell's by introducing the ether as dielectric in the manner indicated by v. Helmholtz.